# INVESTIGATIONS IN CERTAIN-GENERALIZED SPECIAL FUNCTIONS AND FRACTIONAL CALCULUS OPERATORS WITH APPLICATIONS

A

Thesis

Submitted to the

**UNIVERSITY OF KOTA**

in the Partial Fulfillment of the Requirements for the

Award of the Degree of

**DOCTOR OF PHILOSOPHY**

in

**Mathematics**

under the

**Faculty of Science**

*Submitted By*

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**KOTA (RAJASTHAN)-324 005, INDIA**

**January, 2026**

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# *CERTIFICATE*

I feel great pleasure in certifying that the Ph.D. thesis **"INVESTIGATIONS IN CERTAIN-GENERALIZED SPECIAL FUNCTIONS AND FRACTIONAL CALCULUS OPERATORS WITH APPLICATIONS"** submitted by **Mr. Manoj Kumar Tatwal** to the University of Kota in the partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy in mathematics, is based upon her original research work carried out under my guidance.

He has completed the following requirements as per UGC Regulations and research ordinance of the University:

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d) Presentation of research work before the Departmental Committee.
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f) Two paper presentations in the Conferences/ Seminars.

I recommend the submission of the Ph.D. thesis and certify that it is fit to be evaluated by the examiners.

Date: **Prof. (Dr.) B.B.Jaimini**
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# *DECLARATION*

I, **Manoj Kumar Tatwal** hereby certify that the research work presented in my Ph.D. thesis entitled **"INVESTIGATIONS IN CERTAIN-GENERALIZED SPECIAL FUNCTIONS AND FRACTIONAL CALCULUS OPERATORS WITH APPLICATIONS"** which is carried out by me under the supervision of **Prof. (Dr.) B.B. Jaimini** and submitted in the partial fulfilment of the requirement for the award of the degree of Doctor of Philosophy in Mathematics of the University of Kota, represents my ideas in my own words and where others ideas or words have been included in this thesis, I have adequately cited and referenced the original sources.

The work presented in this thesis has not been submitted elsewhere for the award of any degree or diploma from any other institution or university in India or abroad. I declare that I have adhered to all the principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea / data / fact / source in my submission.

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Place: Kota (Research Supervisor)

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| Paper/Submission ID | 5098600 |
| Submitted by | drannakaushik@uok.ac.in |
| Submission Date | 2025-12-28 23:58:43 |
| Total Pages, Total Words | 172, 3038 |
| Document type | Thesis |

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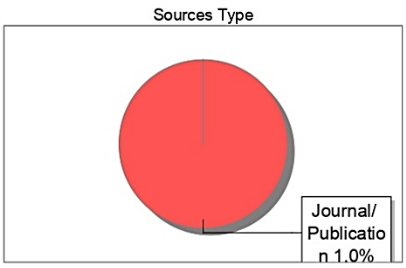


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*This Thesis is Dedicated*

*to the fond memory of*

*My Father*

*Late Shri Heera Lal Tatwal*

# *ACKNOWLEDGEMENT*

I am immensely grateful and indebted to my esteemed guide **Prof. B. B. Jaimini**, M.Sc., M. Phil, Ph.D., Professor, Department of Mathematics, Government College, Kota for his scholarly guidance, academic excellence and supervision throughout the period of my research work.

I wish to express my gratitude for **Prof. Pratima Shrivastava**, Principal, Government College, Kota for providing me all the necessary facilities during the present investigations.

I also wish to express my heartfelt gratitude towards **Prof. Arun Kumar**, Head of Department of Mathematics and faculty members of Department for their constant encouragement and invaluable suggestions provided to me during the entire period of the present study.

I would like to express my sincere thanks to **Prof. Suman Sharma**, Professor in Chemistry, Government College, Kota for her continuous inspiration and encouragement throughout the tenure of this work.

I gratefully acknowledge the support and cooperation received from **Dr. Anna Kaushik**, Deputy Librarian, University of Kota, Kota.

I am eternally grateful to my loving mother **Smt. Dhan Bai Tatwal**. Her love and confidence in me have encouraged me to go ahead in my study and career. My thanks equally go to my Parents-in–law, **Shri Kailash Bairwa** and **Smt. Kaushalya Devi** for invaluable motivation.

My brother **Dr. Deepak Kumar Tatwal** deserves deepest appreciation for his practical assistance, support and encouragement at every step and special thanks to my little sister **Hemlata Tatwal**, for her loving care as of mother, allowing me to focus entirely on my research.

It is hard to express in words; how much I am indebted to My wife **Mrs. Nisha** for providing calming moments and moral support during every hardship of this research. I am so appreciative for her constant love, understanding and encouraging, for her taking up the responsibilities to our family. Endless gratitude to my loving children, **Hridyansh** and **Sanvika**, who turned my stress into playtime adventures, your smiles and playful distractions were the light that guided me to this finish line. You are the constants in my life's most beautiful equation.

Special appreciation to **Lakhan Singh Shakya** for expert typing assistance and consistent help in preparing my documents.

Date:
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**Manoj Kumar Tatwal**
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# TABLE OF CONTENTS

# CHAPTER-1

# INTRODUCTION AND PRELIMINARIES
# TO THE FIELD OF STUDY

This chapter is preliminary and provides the development to the field of Study.

## 1.1 INTRODUCTION

This chapter provides the introduction and preliminaries and updates the advancements in special functions and fractional calculus.

It includes definitions of hypergeometric functions of one and multi variables, Zeta functions along with their generalizations in their series and integral forms, Mittag-Leffler functions and their extensions, related generalized functions and multi variable analogues, the Laplace transform with its properties, generating functions, as well as fractional integral and differential operators. The elementary Gamma and Beta functions are also defined here with their k-generalized forms.

This chapter further offers a concise review of key contributions from early researchers in the direction to unify the study of Mittag-Leffler function and Zeta functions as certain class of functions. The kinetic equation of fractional order are also attracting the researchers in recent years. The generalization of these equations for arbitrary order is also presented here. Additionally, it provides a glossary defining all symbols employed throughout the thesis, along with chapter-by-chapter summaries of the subsequent four chapters.

## 1.2 SPECIAL FUNCTIONS

Special function is a branch of mathematics which has attracted the attention of many pioneer mathematicians namely Euler (1729), Gauss (1812), Clausen (1828), Kummer (1836), Appell (1880) Lauricella (1893) and others. We present briefly the definitions of some relevant functions here in continuation of the developments and closely related to the field of present study.

**(i) Hypergeometric Functions**

**(a) Gauss hypergeometric function**

The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined in the unit disk as follows [96, p. 18, eq. (17)]:

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \qquad ...(1.2.1)$$

where $|z|<1;\ \mathrm{a,b}\in\mathrm{C};\ \mathrm{c}\neq 0,-1,-2,...;$ and $(a)_n$ is Pochhammer symbol defined for $a\in C$ and non-negative integer $n\in N_0$ by

$(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)...(a+n-1)$ ...(1.2.2)

The series given in (1.2.1) is absolutely convergent for $|z|<1$ and for $|z|=1$, when $\mathrm{Re}(c-a-b)>0$, while it is conditionally convergent for $|z|=1$ $(z \neq 1)$ if $-1<\mathrm{Re}(c-a-b)\leq 0$ and it is divergent for $|z|=1$, if $\mathrm{Re}(c-a-b)\leq -1$.

The function ${}_2F_1$ includes as its special cases, the Legendre functions, the incomplete beta function, the complete elliptic function of first and second kind and most of classical orthogonal polynomials.

**(b) The Kummer's confluent hypergeometric function**

If in (1.2.1) we replace $z$ by $\frac{z}{b}$, and let $b\to\infty$ then in view of $\left\{\frac{(b)_n}{b^n} z^n \to z^n\right\}$

we arrive at the well-known Kummer's confluent hypergeometric function

$$\phi(a;c;z) = {}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)^n} \frac{z^n}{n!} \qquad ...(1.2.3)$$

where $a, z \in \mathrm{C};\ \mathrm{c} \neq 0,-1,-2,...;$ but, in contrast to the hypergeometric series in (1.2.1) this series is convergent for any $z \in C$.

**(c) Generalized hypergeometric function**

In the Gaussian hypergeometric series ${}_2F_1(a,b;c;z)$, there are two numerator parameters $a, b$ and one denominator parameter c. A natural generalization of this series is defined and represented by [96, p. 19, eq. (23)] (see also [81]):

$${}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p; \\ \beta_1,...,\beta_q;\end{matrix} z\right] = {}_pF_q(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n,...,(\alpha_p)_n}{(\beta_1)_n,...,(\beta_q)_n} \frac{z^n}{n!} \qquad ...(1.2.4)$$

$$\beta_j \neq 0,-1,-2,...;\ j=1,...,q \qquad ...(1.2.5)$$

(a) if $p \leq q$, the series in (1.2.4) converges for all finite value of $z$,

(b) if $p = q+1$, the series converges for $|z|<1$ and diverges $|z|>1$,

(c) if $p > q+1$, the series never converges except when $z=0$ and the function is only defined when the series terminates.

Furthermore, if we take

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \qquad ...(1.2.6)$$

It is known that the ${}_pF_q$ series, with $p = q+1$, is:

(i) absolutely convergent for $|z| = 1$, if $\mathrm{Re}(\omega) > 0$

(ii) conditional convergent for $|z| = 1, z \neq 1$,if $-1 < \mathrm{Re}(\omega) \leq 0$ and

(iii) divergent for $|z| = 1$if $\mathrm{Re}(\omega) \leq -1$

**(d) Wright's generalized hypergeomtric function**

Further generalization of the series ${}_pF_q$ is due to Fox [22] and Wright [115] [114] who studied the asymptotic expansion of the generalized hypergeometric function defined in the following form

$${}_p\psi_q\left[\begin{matrix}(\alpha_1, A_1),...,(\alpha_p, A_p); \\ (\beta_1, B_1),...,(\beta_q, B_q);\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!} \qquad \ldots(1.2.7)$$

where the coefficients $A_1,...,A_p$ and $B_1,...,B_q$, are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$$

The function defined in (1.2.7) is known as the Wright's generalized hypergeometric function.

In literature Wright's generalized hypergeometric function ${}_p\psi_q[.]$ is also defined and represented in the following manner

$${}_p\psi_q^*\left[\begin{matrix}(\alpha_1, A_1),...,(\alpha_p, A_p); \\ (\beta_1, B_1),...,(\beta_q, B_q);\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)_{A_j n}}{\prod_{j=1}^{q} \Gamma(\beta_j)_{B_j n}} \frac{z^n}{n!} \qquad \ldots(1.2.8)$$

If we take $A_j = 1(j = 1,...,p)$ and $B_j = 1(j = 1,...,\mathrm{q})$ in (1.2.7) the Wright's generalized hypergeometric function reduces into the generalized hypergeometric function ${}_pF_q[.]$ i.e.

$$ {}_p\psi_q\left[\begin{matrix}(\alpha_1,1),\dots,(\alpha_p,1);\\(\beta_1,1),\dots,(\beta_q,1);\end{matrix}\, z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p;\\\beta_1,\dots,\beta_q;\end{matrix}\, z\right] \qquad \text{...(1.2.9)}$$

**(ii) Hypergeometric Functions of Multivariables**

**(a) The Kampé de Fériet function**

In 1880, Appell introduced four functions, $F_1, F_2, F_3$ and $F_4$, which extend the Gauss hypergeometric function ${}_2F_1$ in two variables and are widely referred to as Appell functions today. Kampé de Fériet [96, p. 27, eqs. (28), (29)] later expanded these functions $F_1$ to $F_4$ and their confluent variants.

$$F^{u:v,p}_{w:t,q}\left[\begin{matrix}(a_u):(c_v);(e_p);\\(b_w):(d_t);(f_q);\end{matrix}\, x,y\right]$$

$$=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{n}(a_j)_{m+n}\prod_{j=1}^{v}(c_j)_m\prod_{j=1}^{p}(e_j)_n}{\prod_{j=1}^{w}(b_j)_{m+n}\prod_{j=1}^{t}(d_j)_m\prod_{j=1}^{q}(f_j)_n}\frac{x^m}{m!}\frac{y^n}{n!} \qquad \text{...(1.2.10)}$$

The series is more compact than the one used originally by Kamp de Fériet. The multivariable extension of the series defined in (1.2.10) is also available in the literature and known as Kampé de Fériet hypergeometric function of several variables. The Kampé de Fériet function of several variables is defined and represented by the following form [96, p.38, eqs. (24)-(28)] (see also [106, p. 1127, eq. (4.1)]).

$$F^{p;p_1,\dots,p_n}_{q;q_1,\dots,q_n}\begin{bmatrix}x_1\\x_n\end{bmatrix}\equiv F^{p;p_1,\dots,p_n}_{q;q_1,\dots,q_n}\left[\begin{matrix}(a_p);(\alpha'_{p_1}),\dots,(\alpha'_{p_n});\\(b_q);(\beta'_{q_1}),\dots,(\beta'_{q_n});\end{matrix}\, x_{1,\dots,}x_n\right]$$

$$=\sum_{s_{1,\dots,}s_n=0}^{\infty}\wedge(s_{1,\dots,}s_n)\ \frac{x_1^{s_1}}{s_1!},\dots,\frac{x_n^{s_n}}{s_n!} \qquad \text{…(1.2.11)}$$

where

$$\wedge(s_{1,\dots,}s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\dots+s_n}\prod_{j=1}^{p_1}(\alpha'_j)_{s_1}\dots\prod_{j=1}^{p_n}(\alpha_j^{(n)})_{s_n}}{\prod_{j=1}^{q}(b_j)_{s_1+\dots+s_n}\prod_{j=1}^{q_1}(\beta'_j)_{s_1}\dots\prod_{j=1}^{q_n}(\beta_j^{(n)})_{s_n}} \qquad \text{…(1.2.12)}$$

and for convergence of the multiple hypergeometric series in (1.2.11)

$$1+q+q_k-p-p_k \geq 0, \quad k=1,2,...,n \qquad \text{...(1.2.13)}$$

The equality holds when in addition, either

$$p > q \ and \ |x_1|^{1/p-q} + ... + |x_n|^{1/p-q} < 1 \qquad \text{...(1.2.14)}$$

or

$$p \leq q \ and \ \max\{|x_1| \ + ... + |x_n|\} < 1 \qquad \text{...(1.2.15)}$$

**(iii) Zeta Functions and Related Generalized Functions**

A general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in the following manner [19, p. 27, eq.1.11 (1)] (see also [98, p. 121]):

$$\phi(z,\mathrm{s},a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \qquad \text{...(1.2.16)}$$

$$\left(a \in C/Z_0^-, \mathrm{s} \in C \text{ when} |z| < 1 \text{ and } \mathrm{Re}(s) > 1 \text{ when } |z| = 1\right)$$

and equivalent in the following integral form [19, p.27, eq. 1.11(3)):

$$\phi(z,\mathrm{s},a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1-ze^{-t}} dt \qquad \text{...(1.2.17)}$$

$$\left(\mathrm{Re}(a) > 0; \mathrm{Re}(s) > 0 \text{ when } |z| \leq 1 (z \neq 1); \ \mathrm{Re}(s) > 1 \text{ when } z = 1\right)$$

The general Hurwitz-Lerch Zeta function $\phi(z,\mathrm{s},a)$ defined by (1.2.16), contains, as its special cases, the Riemann Zeta function $\xi(s)$, the Hurwitz (or generalized) Zeta function $\xi(s,a)$ and the Lerch Zeta function $l_s(\xi)$ defined by (see for details, [19, Chapter -I] and [98, Chapter -2])

$$\xi(s) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} = \phi(1,\mathrm{s},1) \qquad \left(\mathrm{Re}(s) > 1\right) \qquad \text{...(1.2.18)}$$

$$\xi(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad \left(\mathrm{Re}(s) > 1; a \in C/Z_0^-\right) \qquad \text{...(1.2.19)}$$

and

$$l_s(\xi) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+1)^s} = \phi\left(e^{2\pi i\xi}, \mathrm{s}, 1\right) \quad \left(\mathrm{Re}(s) > 1; \xi \in R\right) \qquad \text{...(1.2.20)}$$

respectively.

A generalization of (1.2.16) is introduced in series representation by Goyal and Laddha [28, p. 100, eq. (1.5)] as follows:

$$\phi_{\alpha}^{*}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\frac{z^n}{(a+n)^s} \qquad ...(1.2.21)$$

$$\left(a\geq 1,\ \operatorname{Re}(a)>0,\ |z|<1\right)$$

and equivalently in the integral form [28.p. 100, eq. (1.6)]:

$$\phi_{\alpha}^{*}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{\left(1-ze^{-t}\right)^{\alpha}}dt \qquad ...(1.2.22)$$

$\left(\alpha\geq 1, \operatorname{Re}(a)>0 \text{ and either } |z|\leq 1\ (z\neq 1); \operatorname{Re}(s)>0 \text{ or } |z|=1, \operatorname{Re}(s)>1\right)$

for $\alpha=1$, (1.2.21) concides with (1.2.16).

Another generalization of (1.2.16) is introduced and studied by Garg et al. as follows [24, p.27, eq. (1.4)]:

$$\phi_{\alpha,\beta,\gamma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\, n!}\frac{z^n}{(a+n)^s} \qquad ...(1.2.23)$$

$\left(\gamma, a\neq 0,-1,-2,\ldots;\ s\in C \text{ when } |z|<1 \text{ and } \operatorname{Re}(\gamma+s-\alpha-\beta)>0 \text{ when } |z|=1\right)$

and equivalently in the integral form [24, p. 27, eq. (1.5)]:

$$\phi_{\alpha,\beta,\gamma}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-at}\,{}_2F_1\left(\alpha,\beta;\gamma;ze^{-t}\right)dt \qquad ...(1.2.24)$$

$\left(\operatorname{Re}(a)>0; \gamma\neq 0,-1,-2,\ldots;\ \operatorname{Re}(s)>0 \text{ and } |z|<1 \text{ or } |z|=1 \text{ with } \operatorname{Re}(\gamma-\alpha-\beta)>0\right)$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ is the Gauss's hypergeometric function and $(\lambda)_n$ denotes the Pochhammer symbol.

An another extension of general Hurwitz-Lerch Zeta function is also introduced and represented by Lin and Srivastava in the following manner [58, p.727, eq. (8)]:

$$\phi_{\mu,v}^{(\rho,\sigma)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(v)_{\sigma n}}\frac{z^n}{(a+n)^s} \qquad ...(1.2.25)$$

$\left(\mu\in C, a, v\in C/Z_0^-;\ \rho,\sigma\in R^+; \sigma>\rho \text{ when } s,z\in C;\ \sigma=\rho \text{ and } s\in C \text{ when}\right.$

$\left.|z|<1;\ \sigma=\rho \text{ and } \operatorname{Re}(s-\mu+v)>1 \text{ when } |z|=1\right)$

and equivalently in the integral form [58, p. 727, eq. (14)]:

$$\phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,{}_2\psi_1^*\left[\begin{matrix}(\mu,\rho),(1,1); \\ (\nu,\sigma) \quad ;\end{matrix} ze^{-t}\right]dt \qquad ...(1.2.26)$$

$\left(\mathrm{Re}(a)>0, \mathrm{Re}(s)>0 \text{ when } |z|\le 1 (z\ne 1);\ \mathrm{Re}(s)>1 \text{ when } z=1\right)$.

To unify all the scattered definitions of Zeta functions and its generalizations mentioned as above Srivastava et al. defined the extended Hurwitz-Lerch Zeta function $\phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(z,s,a)$ in the following series form [101, p.491, eq. (1.20)]:

$$\phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}\,n!}\frac{z^n}{(a+n)^s} \qquad ...(1.2.27)$$

$\left(\lambda,\mu\in C; a,\nu\in C/Z_0^-; \rho,\sigma,k\in R^+; k-\rho-\sigma>-1 \text{ when } s,z\in C; k-\rho-\sigma=-1 \text{ and }\right.$

$s\in C$ when $|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}k^k$; $k-\rho-\sigma=-1$ and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when

$\left.|z|=\delta^*=\rho^{-\rho}\sigma^{-\sigma}k^k\right)$

and equivalently in the integral form [101, p.494, eq. (2.4)]:

$$\phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,{}_2\psi_1^*\left[\begin{matrix}(\lambda,\rho)(\mu,\sigma); \\ (\nu,\mathrm{k}) \quad ;\end{matrix} ze^{-t}\right]dt \qquad ...(1.2.28)$$

$\left(\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0, \rho,\sigma,k\in R^+ \text{ and } k-\rho-\sigma\ge -1 \text{ (with equality when}\right.$

$\left.\left.|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}k^k\right)\right)$.

A generalization of extended Hurwitz-Lerch Zeta function $\phi_{\lambda,\mu,\nu}^{\rho,\sigma,k}(z,s,a)$ is defined by [101, p.503, eq. (6.2)]:

$$\phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}\;n!}\frac{z^n}{(n+a)^s} \qquad \ldots(1.2.29)$$

$\left(p,q\in N_0; \lambda_j\in C\,(j=1,\ldots,p); a,\mu_j\in C/Z_0^-\,(j=1,\ldots,\mathrm{q})\sigma_j,\rho_k\in R^+\right.$

$\left.(j=1,\ldots,\mathrm{p};\mathrm{k}=1,\ldots,\mathrm{q})\right)$

and equivalently in the integral form [101, p.504, Eq. (6.4)]:

$$\phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,{}_p\psi_q^*\left[\begin{matrix}(\lambda_1,\rho_1),\ldots,(\lambda_p,\rho_p); \\ (\mu_1,\sigma_1),\ldots,(\mu_q,\sigma_q);\end{matrix} ze^{-t}\right]dt \qquad ...(1.2.30)$$

$\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0$.

A class of functions related to general Hurwitz-Lerch Zeta function $\theta_{\mu}^{\lambda}(x,\alpha,a,b)$ and its multivariable analogue are also available in the literature these classes are defined and represented respectively in the following forms [80, p. 90, eq. (1.7); p. 99, eq. (5.2)]:

$$\theta_{\mu}^{\lambda}(x,\alpha,a,b)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{\alpha-1}e^{-at-bt^{-\lambda}}\left(1-xe^{-t}\right)^{-\mu}dt \qquad ...(1.2.31)$$

$\lambda>0,\mu\geq 1,\mathrm{Re}(a)>\mathrm{Re}(b)>0$ when $\mathrm{Re}(b)=0$ then either $|x|\leq 1\ (x\neq 1),\mathrm{Re}(\alpha)>0$ or $x=1,\mathrm{Re}(\alpha))$;

and

$$\theta_{(p_i)}^{(\lambda,\mu_1)}(x_1,...,x_n,\alpha,a,b)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{\alpha-1}e^{-at-bt^{-\lambda}}\prod_{i=1}^{n}\left(1-x_ie^{-p_it}\right)^{-\mu_i}dt \qquad ...(1.2.32)$$

$\left(\lambda>0,\mu_i\geq 1,\mathrm{Re}(p_i)>0(i=1,2,...,n)\mathrm{Re}(a)>\mathrm{Re}(b)>0\right.$; when $b=0$ then either $\max\limits_{1\leq i\leq n}(|x_i|)<1\ (x_i\neq 1)$, $\mathrm{Re}(a)>0$ or $x_i=1(i=1,2,...,n)$, $\left.\mathrm{Re}(\alpha)>\max\limits_{1\leq i\leq n}(\mu_i)\right)$

The series representation of the class of functions related to general Hurwitz-Lerch Zela function defined in (1.2.31) and (1.2.32) are respectively as follows [80, p. 91, eq. (2.1); p. 98, eq. (5.1)]:

$$\theta_{\mu}^{\lambda}(x,\alpha,a,b)=\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha-\lambda m)(-b)^m}{m!}\sum_{n=0}^{\infty}\frac{(\mu)_n}{(a+n)^{\alpha-\lambda m}}\frac{x^n}{n!} \qquad ...(1.2.33)$$

$\left(\lambda>0,\ \mu\geq 1,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(b)\geq 0,\mathrm{Re}(\alpha)\neq v\lambda\ (v\in N),\ |x|\leq 1\right)$;

$$\theta_{(\rho_1)}^{(\lambda,\mu_1)}(x_1,...,x_n,\alpha,a,b)$$

$$=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha-\lambda k)(-b)^k}{\mathrm{k}!}\sum_{m_1,...,m_n=0}^{\infty}(a+\Omega)^{-(\alpha-\lambda k)}\prod_{i=1}^{n}\left\{\frac{(\mu_i)_{m_i}\ x_i^{m_i}}{m_i\,!}\right\} \qquad ...(1.2.34)$$

$(\lambda>0,\mu_i\geq 1(i=1,2,...,n),\ \mathrm{Re}(a)>0,\ \mathrm{Re}(b)\geq 0,$

$\mathrm{Re}(\alpha)\neq v\lambda(v\in N),\ \max\limits_{1\leq i\leq n}(|x_i|)\leq 1,\ \Omega=\sum\limits_{i=1}^{n}p_im_i$ ).

Furthermore, Gupta [30, p. 110, eqs. (2.6.1), (2.6.2)] also consider a new generalization of the extended Hurwitz-Lerch Zeta function, as defined in equations

(1.2.27) and (1.2.28), in both single-variable and multi-variable forms which are defined and represented as follows:

$$\phi_{\lambda,\mu,\nu}^{(\alpha,\rho,\sigma,k)}\left(z,s,a,b\right)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at-bt^{-\alpha}}\;{}_2\psi_1^*\left[\begin{matrix}(\lambda,\rho)(\mu,\sigma); \\ (v,\mathrm{k}) \quad ;\end{matrix} ze^{-t}\right]dt \qquad ...(1.2.35)$$

($\alpha>0,\;\mathrm{Re}(b)>0,\;\lambda,\mu\in C,\;a,v\in C/Z_0^-\;\;\min\{\mathrm{Re}(a),\;\mathrm{Re}(s)\}>0,\;\rho,\sigma,k\in R^+$

and $k-\rho-\sigma>-1$ (with equality when $|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}k^k$ )

and

$$\phi_{(\lambda_j^{(i)},v_k^{(i)},\beta_i)}^{(\alpha,\rho_j^{(i)},\sigma_k^{(i)})}\left(z_{1,\dots,}z_r;s,a,b\right)$$

$$=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at-bt^{-\alpha}}\prod_{i=1}^r\;{}_{p_i}\psi_{q_i}^*\left[\begin{matrix}\left(\lambda_{p_i}^{(i)},\rho_{p_i}^{(i)}\right); \\ \left(v_{q_i}^{(i)},\sigma_{q_i}^{(i)}\right);\end{matrix} z_ie^{-\beta_i t}\right]dt \qquad ...(1.2.36)$$

Where for $\forall\; i=1,2,...,r;\;\beta_i\equiv\beta_{1,\dots,}\beta_r\;;\;\rho_j^{(i)}\equiv\rho_1^{(1)},...,\rho_{p_i}^{(i)}\;;\;\lambda_j^{(i)}\equiv\lambda_1^{(1)},...,\lambda_{p_i}^{(i)}$

$\sigma_k^{(i)}\equiv\sigma_1^{(1)},...,\sigma_{q_i}^{(i)}\;;\;v_k^{(i)}\equiv v_1^{(1)},...,v_{q_i}^{(i)}$ *and* $\alpha>0,\;\mathrm{Re}(b)>0,\;\min\{\mathrm{Re}(a),\;\mathrm{Re}(s)\}>0,$

$p_i,q_i\in\;N_0;\;\lambda_j^{(i)},\beta_i,z,s\in C;\;(j=1,...,p_i);\;a,v_k^{(i)}\in C/Z_0^-;\;(k=1,...,q_i);$

$\rho_j^{(i)},\sigma_k^{(i)}\in R^+\;(j=1,...,p_i;k=1,...,q_i)$.

The series representation of new generalized Hurwitz-Lerch Zeta function defined in (1.2.35) and (1.2.36) are respectively as follows [30, p. 113, eq. (2.7.1); p. 114, eq. (2.7.3)]:

$$\phi_{\lambda,\mu,\nu}^{(\alpha,\rho,\sigma,k)}\left(z,s,a,b\right)=\frac{1}{\Gamma(s)}\sum_{m=0}^\infty\frac{(-b)^m\,\Gamma(s-\alpha m)}{m!}\sum_{n=0}^\infty\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(v)_{kn}(a+n)^{s-\alpha m}}\frac{x^n}{n!} \qquad \ldots(1.2.37)$$

$(\alpha>0,\;\mathrm{Re}(a)>0,\;\mathrm{Re}(b)>0,\;\lambda,\mu\in C,\;a,v\in C/Z_0^-,\;\rho,\sigma,k\in R^+,$

$\mathrm{Re}(s)\neq\alpha d(d\in N),\;|z|\le 1;\;s,z\in C)$

and

$$\phi_{(\lambda_j^{(i)},v_k^{(i)},\beta_i)}^{(\alpha,\rho_j^{(i)},\sigma_k^{(i)})}\left(z_{1,\dots,}z_r;s,a,b\right)=\frac{1}{\Gamma(s)}\sum_{m=0}^\infty\frac{(-b)^m\,\Gamma(s-\alpha m)}{m!}$$

$$\sum_{n_1,\dots,n_r=0}^\infty\left(a+\sum_{i=1}^r n_i\beta_i\right)^{-(s-\alpha m)}\prod_{i=1}^r\left[\frac{\prod_{j=1}^{p_i}\left(\lambda_j^{(i)}\right)_{\rho_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(v_k^{(i)}\right)_{\sigma_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i\,!}\right] \qquad \ldots(1.2.38)$$

For $\forall\ i=1,2,...,r;\ \alpha>0;\ \mathrm{Re}(b)\geq 0;\ p_i,q_i\in N_0;\ \rho_j^{(i)}\in R^+,\ \lambda_j^{(i)}\in C\ (j=1,...,p_i);$ the $a,v_k^{(i)}\in C/Z_0^-;\ \sigma_k^{(i)}\in R^+;\ (k=1,...,q_i);\ \min\{\mathrm{Re}(a),\ \mathrm{Re}(s)\}>0;\ \mathrm{Re}(s)\neq\alpha d(d\in N)$ $\max_{1\leq i\leq r}(|z_i|)\leq 1$.

**(iv) Mittag-Leffler Function and Related Generalized Functions**

**(a) Classical Mittag-Leffler function**

The entire function of the form

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\qquad (z\in C;\ \mathrm{Re}(\alpha)>0)\qquad ...(1.2.39)$$

was introduced by Mittag-Leffler [66] and is therefore known as the Mittag-Leffler function. The elementry properties of this function were studied by Mittag-Leffler ([66], [67]) and by Wiman [113].

A generalization of Mittag-Leffler function defined in (1.2.39) was studied by Wiman [113] in the following form:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}\qquad ...(1.2.40)$$

where $\alpha,\beta\in \mathrm{C},\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,z\in C$. Its properties are similar to those of Mittag -Leffler function defined in (1.2.39). Its basic properties were investigated by Agarwal [1] Humbert [41], Humbert and Agarwal [40] and Dzherbashyan [18].

When $\beta=1,\ E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function (1.2.39).

i.e.

$$E_{\alpha,1}(z)=E_\alpha(z)\qquad (z\in C,\mathrm{Re}(\alpha)>0)\qquad ...(1.2.41)$$

More detailed information may be found in the book by Erdelyi et al. ([21], Vol. 3, section 18.1) and Dzherbashyan ([18], chapter -III).

**(b) Generalized Mittag-Leffler function**

A generalization of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined in (1.2.40) was introduced by Prabhakar [78] as follows:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}\qquad ...(1.2.42)$$

where $\alpha, \beta, \gamma \in \mathrm{C}$, $\mathrm{Re}(\alpha) > 0$ and $(\gamma)_k$ is the well-known Pochhammer symbol defined in (1.2.2). This function is an entire function of $z$ of order $[\mathrm{Re}(\alpha)]^{-1}$. Its properties were investigated by Prabhakar [78] and recently by Kilbas and Saigo [49]. In particular, when $\gamma = 1$, it coincides with the Mittag-Leffler function (1.2.40) i.e.

$$E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z) \qquad (z, \beta \in C,\ \mathrm{Re}(\alpha) > 0) \qquad \text{...(1.2.43)}$$

when $\alpha = 1$, $E_{1,\beta}^{\gamma}(z)$ coincides with the Kummer confluent hypergeometric function $\phi(\gamma;\beta;z)$, apart from a constant factor $[\Gamma(\beta)]^{-1}$, i.e.

$$E_{1,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\beta)} \phi(\gamma;\beta;z) \qquad \text{...(1.2.44)}$$

A generalization of (1.2.42) has been studied by Shukla and Prajapati [93] in the following form:

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n + \beta)} \frac{z^k}{n!} \qquad \text{...(1.2.45)}$$

where $\alpha, \beta, \gamma \in \mathrm{C}, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0; q \in (0,1) \cup N$.

Another generalization of Mittag-Leffler function defined in (1.2.40) and (1.2.43) was introduced and studied by Srivastava and Tomvoski [97] in the following form:

$$E_{\alpha,\beta}^{\gamma,K}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nK}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \qquad \text{...(1.2.46)}$$

where $\alpha, \beta, \gamma, K \in \mathrm{C}, \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(K) - 1\}, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(K) > 0$.

A multi-parameter Mittag-Leffler function introduced and studied by Salim and Faraj [86, p.2, eq. (6)] in the following form:

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}} \qquad \text{...(1.2.47)}$$

where $\alpha, \beta, \gamma, \delta \in C;\ \min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(\delta)\} > 0$ ; $p, q > 0$ and $q \le \mathrm{Re}(\alpha) + p$.

Further generalization of multi-parameter Mittag-Leffler function is investigated and defined by Khan and Ahmed [47, p.2, eq. (1.9)] as follows:

$$E_{\alpha,\beta,v,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} (\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(v)_{\sigma n} (\delta)_{pn}} \qquad \text{...(1.2.48)}$$

where $\alpha,\beta,\gamma,\delta,\mu,v,\rho,\sigma \in C;\ \ p,q>0\ ;q\le \operatorname{Re}(\alpha)+p$, and

$\min\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\gamma),\operatorname{Re}(\delta),\operatorname{Re}(\mu),\operatorname{Re}(v),\operatorname{Re}(\rho),\operatorname{Re}(\sigma)\}>0.$

At $\mu=v$ *and* $\rho=\sigma$, it reduce to eq. (1.2.47) and if $\mu=v,\rho=\sigma$ and $p=\delta=1$, then it reduce to eq. (1.2.45), in addition of that if $q=1$ we get eq.(1.2.42).

A multi-index Mittag-Leffler function defined and studied by Kiryakova [52, p.1888, eq. (7)] in the following form

$$E_{\left(\frac{1}{\alpha_i}\right),(\beta_i)}(z)=E^{(m)}_{\left(\frac{1}{\alpha_i}\right),(\beta_i)}(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\beta_1+r/\alpha_1)....(\beta_m+r/\alpha_m)} \qquad ...(1.2.49)$$

where $z,\alpha_i,\beta_i\in \mathrm{C},\operatorname{Re}(\alpha_i)>0,\ i=1,...,m$.

Another generalization of the function defined in (1.2.42) and (1.2.47) is defined by [48, p.47, eq. (1.9.14)]:

$$E_\rho\left(\left(\alpha_j,\beta_j\right)_{1,m};z\right)=\sum_{k=0}^{\infty}\frac{(\rho)_k}{\prod_{j=1}^{m}\Gamma\left(\alpha_j k+\beta_j\right)}\frac{z^k}{k!}$$

$$=\frac{1}{\Gamma(\rho)}\,{}_1\psi_m\left[\begin{matrix}(\rho,1) & ; \\ (\beta_1,\alpha_1),...,(\beta_m,\alpha_m); & \end{matrix}\ z\right] \qquad ...(1.2.50)$$

where $z,\rho,\beta_j\in \mathrm{C},\operatorname{Re}(\alpha_j)>0,\ j=1,...,\mathrm{n}$ and $m\in N$.

To extend the differential property of Mittag–Leffler functions defined in (1.2.42) and (1.2.48), Kilbas and Saigo [49] define the following function

$$E_{\alpha,m,l}(z)=\sum_{k=0}^{\infty}c_k z^k \qquad ...(1.2.51)$$

with $c_0=1, c_k=\sum_{j=0}^{k-1}\frac{\Gamma\left[\alpha(jm+l)+1\right]}{\Gamma\left[\alpha(jm+l+1)+1\right]} \qquad (k\in N) \qquad ...(1.2.52)$

$\alpha,l\in \mathrm{C},\mathrm{m}\in \mathrm{R}$ Such that $\operatorname{Re}(\alpha)>0,\mathrm{m}>0,\alpha(jm+l)\notin Z^-\ (j\in N_0)$.

This function occurs as the solution of certain Abel-Volterra integral equations. Properties of $E_{\alpha,m,l}(z)$ are investigated by Gorenflo et al. [27], Kilbas and Saigo ([50], [49]), Saigo and Kilbas [85] and others.

**(c) Multivariable Mittag-Leffler function**

The multivariable Mittag-Leffler function $E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n)$ of $n$ complex variables $z_1,\dots,z_n \in C$ with complex parameters $a_1,\dots,a_n,b \in C$ is defined by [48, p.49, eqs. (1.9.27), (1.9.28)]:

$$E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n) = \sum_{k=0}^{\infty} \sum_{l_1,\dots,l_n \geq 0}^{l_1+\dots+l_n=k} \binom{k}{l_1,\dots,l_n} \frac{\prod_{j=1}^{n} z_j^{l_j}}{\Gamma\left(b+\sum_{j=1}^{n} a_j l_j\right)} \quad \dots(1.2.53)$$

in terms of multinomial coefficients:

$$\binom{k}{l_1,\dots,l_n} = \frac{k!}{(l_1)!\dots(l_n)!} \qquad k,l_1,\dots,l_n \in N_0 \quad \dots(1.2.54)$$

A further extension of both the Mittag-Leffler functions defined in (1.2.46) and (1.2.49) was introduced and studied by Saxena and Nishimoto [90, p.44] in the following form:

$$E_{\gamma,K}\left[(\alpha_j,\beta_j)_{1,m};z\right] = E_{\gamma,K}\left[(\alpha_1,\beta_1),\dots,(\alpha_m,\beta_{mj});z\right]$$

$$= \sum_{r=0}^{\infty} \frac{(\gamma)_{rK}}{\prod_{j=1}^{m}\Gamma(\beta_j+\alpha_j r)} \frac{z^r}{r!} \quad \dots(1.2.55)$$

where $z,\gamma,\alpha_j,\beta_j \in C, \sum_{j=1}^{m} \operatorname{Re}(\alpha_j) > \operatorname{Re}(K)-1, j=1,\dots,m, \operatorname{Re}(K)>0$.

A multivariable analogue of generalized Mittag-Leffler function (1.2.42) in the form [88, p.537, eq. (2.1)] (See also (25]):

$$E_{(\rho_j),\lambda}^{(\gamma_j)}(z_1,\dots,z_m) = E_{(\rho_1,\dots,\rho_m),\lambda}^{(\gamma_1,\dots,\gamma_m)}(z_1,\dots,z_m)$$

$$= \sum_{k_1,\dots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1}\cdots(\gamma_m)_{k_m}}{\Gamma\left(\lambda+\sum_{j=1}^{m} p_j k_j\right)} \frac{z_1^{k_1}\dots z_m^{k_m}}{(k_1)!\dots(k_m)!} \quad \dots(1.2.56)$$

with $\lambda,\gamma_j,\rho_j \in C$ and $\operatorname{Re}(\rho_j)>0, j=1,\dots,m$.

A new generalization of multivariable Mittag-Leffler functions defined in (1.2.56) was introduced by Gurjar et al. [33, p.7, eq. (1.8)] in the following form:

$$E_{(\rho_j;\lambda;q_j)}^{(\gamma_j;l_j;p_j)}(z_1,...,z_m)=E_{\rho_1,...,\rho_m;\lambda;q_1,...,q_m}^{\gamma_1,...,\gamma_m;l_1,...,l_m;p_1,...,p_m}(z_1,...,z_m)$$

$$=\sum_{r_1,...,r_m=0}^{\infty}\frac{(\gamma_1)_{p_1r_1}...(\gamma_m)_{p_mr_m}}{\Gamma\left(\lambda+\sum_{j=1}^{m}\rho_j r_j\right)}\frac{z_1^{r_1}...z_m^{r_m}}{(l_1)_{q_1r_1}...(l_m)_{q_mr_m}} \qquad ...(1.2.57)$$

where $\lambda,\rho_j,\gamma_j,l_j\in C;\ \min_{1\le j\le m}\{\mathrm{Re}(\lambda),\mathrm{Re}(\rho_j),\mathrm{Re}(\gamma_j),\mathrm{Re}(l_j)\}>0$ and

$p_j,q_j>0;\ p_j<q_j+\mathrm{Re}(\rho_j);\ j=1,2,...,m$.

A generalization of multivariable and multi-index Mittag-Leffler function was introduced and studied by Jaimini et al. [44, p.2, eq. (3)] in the following manner:

$$E_{(\rho_1^{(r)}),...,(\rho_m^{(r)});\beta_1,...,\beta_m}^{(\gamma_r),(l_r)}(z_1,...,z_r)=\sum_{k_1,...,k_r=0}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_{l_ik_i}}{\prod_{i=1}^{m}\Gamma\left(\beta_j+\sum_{i=1}^{r}\rho_j^{(i)}k_i\right)}\frac{z_1^{k_1}}{k_1!},...,\frac{z_r^{k_1}}{k_r!} \qquad ...(1.2.58)$$

where $\beta_j,\gamma_i,l_i,\rho_j^{(i)}\in C;\ \mathrm{Re}(\rho_j^{(i)})>0;\ \mathrm{Re}(l_i)>0;\ \beta_j\notin Z_0^-=\{0,-1,-2,...\};$

$\rho_j^{(r)}=\rho_j^{'},\rho_j^{''},...,\rho_j^{(r)};\ i=1,...,r;\ and\ \ j=1,...,m.$

**(v) Definition of elementary special functions**

**(a) Factorial function**

The factorial function is denoted and defined by Rainville [81] in the following form:

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=\begin{cases}1 & ,(m=0)\\ a(a+1)...(a+m-1), & m\in N\end{cases} \qquad ...(1.2.59)$$

The function $(a)_m$ defined in (1.2.59) is also known as pochhammer symbol.

**(b) Gamma function**

The integral form of Gamma function is defined in [29, p.884, eq.312(2)] as the following form:

$$\Gamma\alpha=\int_0^{\infty}x^{\alpha-1}e^{-x}dx \qquad (\mathrm{Re}(\alpha)>0) \qquad ...(1.2.60)$$

**(c) Beta function**

The beta function of two variables is defined as follows [81]:

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \qquad \ldots(1.2.61)$$

where $\mathrm{Re}(x),\mathrm{Re}(y)>0$

and in terms of gamma function, it is written as

$$B(x,y)=\frac{\Gamma x\ \Gamma y}{\Gamma(x+y)} \qquad \ldots(1.2.62)$$

An extension of Eulers beta function was presented by Choudhry et al. [9] (see also [10]) in the following form:

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt \qquad \ldots(1.2.63)$$

where $\mathrm{Re}(p)>0,\mathrm{Re}(x)>0$ *and* $\mathrm{Re}(y)>0$.

**(vi) k-generalized functions**

Further Diaz and Pariguan [13] introduced and defined the pochhammeer k-symbol $(a)_{n,k}$, k-generalized gamma function $\Gamma_k$ , beta function $B_k$ in the following form

**(a) Pochhammer k-symbol**

For $k>0$, Pochhammer k-symbol is denoted and defined by

$$(a)_{n,k}=\begin{Bmatrix} a(a+k)(a+2k)...(a+(n-1)k); & for\ n\geq 1,\ a\neq 0 \\ 1\ ; & if\ n=0 \end{Bmatrix} \qquad \ldots(1.2.64)$$

**(b) k-Gamma function**

The integral form of k-Gamma function is defined as follows:

$$\Gamma_k(z)=\int_0^\infty t^{z-1}e^{-t^k/k}dx \qquad \ldots(1.2.65)$$

where $z\in C$ and $k>0$.

**(c) k-Beta function**

The k-beta function of two variables is defined as follows:

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{x/k-1}(1-t)^{y/k-1}dt \qquad \ldots(1.2.66)$$

where $\mathrm{Re}(x), \mathrm{Re}(y) > 0$

and in terms of k-gamma function, k-beta function is defined as

$$B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \qquad \ldots(1.2.67)$$

Further Kokologiannaki [54], Kokologiannaki and Krasniqi [53], Krasniqi [55] (see also [63], [69]) have worked on the generalized k-gamma and k-beta function and discussed the following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x) \qquad \ldots(1.2.68)$$

where $x > 0$ *and* $k > 0$.

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} \qquad \ldots(1.2.69)$$

where $n \in N$, $x > 0$ *and* $k > 0$.

$$\Gamma_k(k) = 1, \qquad k > 0 \qquad \ldots(1.2.70)$$

Further an extension of the k-Beta function was introduced as follows [71] (see also [70]):

$$B_k(x, y; p) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} e^{-\frac{p^k}{kt(1-t)}} dt \qquad \ldots(1.2.71)$$

where $k > 0,\ \mathrm{Re}(p) > 0,\ \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$

For particular value $k = 1$, it reduces to (1.2.63) in addition of that if $p = 0$ we get eq. (1.2.61).

### (d) k-Mittag Leffler function

Dorrego and Cerutti [17] introduced a new Mittag-Leffler function. It is defined and represented in the following form:

$$E_{k,l,m}^{\rho}(x) = \sum_{n=0}^{\infty} \frac{(\rho)_{n,k}}{\Gamma_k(nl+m)} \frac{x^n}{n!} \qquad \ldots(1.2.72)$$

where $k > 0$, $x, l, m, \rho \in C; \mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0$, $(\rho)_{n,k}$ and $\Gamma_k(z)$ are defined in (1.2.64), (1.2.65) respectively.

### (e) Mittag Leffler type function

To unify $j-$pseudo-hyperbolic, $j-$pseudo-trigonometric and $\lambda-$hyperbolic function, Pathan and Bin-Saad [75] introduced and studied the following Mittag-Leffler type function of arbitrary order (see also [91]):

$$E_{v,\mu}^{j,k}(z)=\sum_{n=0}^{\infty}\frac{z^{nj+k}}{\Gamma(\mu+v(nj+k))} \quad \ldots(1.2.73)$$

where $j\geq 1, k\geq 0, v,\mu,z\in C; \mathrm{Re}(v)>0, \mathrm{Re}(\mu)>0.$

The study related to this function is rapidly attracting the researchers.

**(vii) Class of functions**

The asymptotic expansion of the Taylor-Maclaurin series was studied by Wright [114, p. 424] in the following form:

$$\in_{\alpha,\beta}(\phi;z)=\sum_{n=0}^{\infty}\frac{\phi(n)}{\Gamma(\alpha n+\beta)}z^n \quad \ldots(1.2.74)$$

($\alpha,\beta\in C$ ; $\mathrm{Re}(\alpha)>0$)

where $\phi(n)$ is a function satisfying suitable conditions.

Barnes [4] was considered the asymptotic expansions of functions in the class it is defined as follows:

$$E_{\alpha,\beta}^{(k)}(s;z)=\sum_{n=0}^{\infty}\frac{z^n}{(n+k)^s\ \Gamma(\alpha n+\beta)} \quad \ldots(1.2.75)$$

($\alpha,\beta\in C$ ; $\mathrm{Re}(\alpha)>0$) and for suitably-restricted parameters $k$ *and* $s$.

At $Lim\ \alpha\to 0$ We have the following relationship

$$\lim_{\alpha\to 0}\left\{E_{\alpha,\beta}^{(k)}(s;z)\right\}=\frac{1}{\Gamma\beta}\Phi(z,s,k) \quad \ldots(1.2.76)$$

where $\Phi(z,s,k)$ is Hurwitz-Lerch Zeta function defined in (1.2.16).

For unification of functions defined in (1.2.74), (1.2.75) and extended Hurwitz-Lerch Zeta function (1.2.29), recently Srivastava [94] introduced a function. it is represented in the following form:

$$\in_{\alpha,\beta}(\phi;z,s,k)=\sum_{n=0}^{\infty}\frac{\phi(n)}{(n+k)^s\,\Gamma(\alpha n+\beta)}z^n \quad \ldots(1.2.77)$$

($\alpha,\beta\in C$ ; $\mathrm{Re}(\alpha)>0$) and $\phi(n)$ is suitably-restricted function.

## 1.3 GENERATING FUNCTIONS

### (i) Linear Generating Functions

Consider a two-variable function $F(x,t)$ which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in $t$ such that:

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x) t^n \qquad \ldots(1.3.1)$$

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of $t$. Then the expansion (1.3.1) of $F(x,t)$ is said to have generated the set $\{f_n(x)\}$ and $F(x,t)$ is called a linear generating function (or simply, a generating function) for the set $\{f_n(x)\}$. If a generating function $\phi(x,t)$ possesses a power series expansion which is obviously divergent whenever $t \neq 0$, we shall find it convenient to use the following notation to indicate divergence.

$$\phi(x,t) \cong \sum_{n=0}^{\infty} \phi_n(x) t^n \qquad \ldots(1.3.2)$$

The foregoing definition may be extended slightly to include a generating function of the type:

$$G(x,t) = \sum_{n=0}^{\infty} c_n g_n(x) t^n \qquad \ldots(1.3.3)$$

where the sequence $\{c_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$ but is independent of $x$ and $t$.

where the sequence $\{\gamma_n\}$ is independent of $x$, $y$ and $t$, the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different, and $\alpha(n)$ and $\beta(n)$ are functions of $n$ which are not necessarily equal.

### (ii) Multivariable Generating Functions

In each of the above definitions, the sets generated are functions of only one variable, suppose now that $G(x_1,\ldots,x_r;t)$ is a function of $r+1$ variables, which has a formal expansion in powers of $t$ such that:

$$G(x_1,\ldots,x_r;t) = \sum_{n=0}^{\infty} c_n g_n(x_1,\ldots,x_r) t^n \qquad \ldots(1.3.4)$$

where the sequence $\{c_n\}$ is independent of the variables $x_1,...,x_r$ and $t$. Then we shall say that $G(x_1,...,x_r;t)$ is a generating function for the set $\{g_n(x_1,...,x_r)\}_{n=0}^{\infty}$ corresponding to the non zero coefficient $c_n$.

### (iii) Multilinear and Multilateral Generating Functions

A multivariable generating function $G(x_1,...,x_r;t)$ given by (1.3.4) is said to be a multilinear generating function if:

$$g_n(x_1,...,x_r) = f_{\alpha_1(n)}(x_1)...f_{\alpha_r(n)}(x_r) \qquad ...(1.3.5)$$

where $\alpha_1(n),...,\alpha_r(n)$ are functions of $n$ which are not necessarily equal.

More generally, if the functions occurring on the right hand side of (1.3.5) are all different, the multivariable generating function in (1.3.4) will be called a multilateral generating function. Furthermore details of generating functions see [99, p. 78].

The more work on generating functions may be found in Szegö [111], Polya and Szegö [77], Rainville [81], Carlitz [7] and Srivastava and Singhal [102].

## 1.4 FRACTIONAL CALCULUS

Fractional calculus presents the study and application of derivatives and integrals of arbitrary order and provides the generalization of classical calculus for integer order. Over the last thirty years, it has become increasingly significant due to its practical use in many areas of science and engineering. In mathematical analysis, there are many fields where fractional calculus have been effectively used in various branches such as integral and differential equations, special functions, integral transform, operational calculus and univalent and multivalent functions. (See [90] [88]).

Several mathematicians have made notable contributions to its development in the last century, included P.S. Laplace, J.B.J. Fourier, A.V. Letnikov, A.P. Laurrent, P.A. Nekrassov, A. Kurg, J. Hadamard, O. Heaviside, S. Pincherie, G.H. Hardy, J.E. Littlewood, H. Weyl, P. Levy, P. Marchand, H.T. Devis, A. Zygmond, A. Erdelyi and H. Kober.

Fractional calculus has its origin in the question of extension of meaning to the familiar differential operator $D^n$. When $n$ is not necessarily positive integer means where it is any arbitrary number rational, irrational or complex.

The first approach to the idea of fractional derivatives was made by Lebnitz (1695), in his later to L' Hopital. Further in (1819), the first mention of derivatives of arbitrary order appeared in the text by French mathematician Lacroix [57]. He started with the formula

$$\frac{d^m x^n}{dx^m} = \frac{n!}{(n-m)!} x^{n-m} \qquad \text{...(1.4.1)}$$

$n, m$ are positive integers.

Between (1832) to (1855) Liouville wrote several papers on fractional derivatives and made the major study of fractional calculus. The papers by Liouville (1832) and (1837) were the first once that contained an application of fractional calculus to the solution of some type of linear ordinary differential equation. In other paper (1835) Liouville considered the effect of change of variable in fractional derivatives and integrals.

In the literature now these fractional integrals and derivatives are known as the Riemann Liouville fractionals integrals and derivatives.

**The Fractional Integrals and Derivatives**

**(i) The Riemann - Liouville operators**

There are many papers, in which the definitions of fractional derivatives and integrals of the function of single variable are reported. Some of them as follows. The Riemann - Liouville fractional integrals and derivatives studied in [62] [61] [65] are defined and represented in the following manner (see also (72],[87]):

The Riemann-Liouville fractional integrals $\left(I_{a+}^{\alpha} f\right)$ and $\left(I_{b-}^{\alpha} f\right)$ of order $\alpha \in C\left(\operatorname{Re}(\alpha) > 0\right)$ are defined as follows:

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}\, dt \qquad \left(x > a \;; \operatorname{Re}(\alpha) > 0\right) \qquad \text{...(1.4.2)}$$

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}\, dt \qquad \left(x < b \;; \operatorname{Re}(\alpha) > 0\right) \qquad \text{...(1.4.3)}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left sided and right sided fractional integrals.

The Riemann-Liouville fractional derivatives $\left(D_{a+}^{\alpha} f\right)$ and $\left(D_{b-}^{\alpha} f\right)$ of order $\alpha \in C\left(\operatorname{Re}(\alpha) \geq 0\right)$ are defined as follows:

$$\left(D_{a+}^{\alpha} f\right) = \left(\frac{d}{dx}\right)^{n} \left(I_{a+}^{n-\alpha} f\right)(x)$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \quad \left(n = \left[\operatorname{Re}(\alpha)\right] + 1; x > a\right) \qquad \text{...(1.4.4)}$$

and

$$\left(D_{b-}^{\alpha} f\right) = \left(-\frac{d}{dx}\right)^{n} \left(I_{b-}^{n-\alpha} f\right)(x)$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^{n} \int_{x}^{b} \frac{f(t)}{(t-x)^{\alpha-n+1}} dt \quad \left(n = \left[\operatorname{Re}(\alpha)\right] + 1; x < b\right) \qquad \text{...(1.4.5)}$$

respectively, where $\left[\operatorname{Re}(\alpha)\right]$ means the integral part of $\operatorname{Re}(\alpha)$.

From among the various definitions of fractional calculus (i.e. fractional derivatives and integrals) given in literature, we choose here the following definition which were used earlier by Owa [74] and many others.

The fractional integral of order $\lambda$ is defined, for a function $f(z)$, by

$$D_{z}^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \qquad \text{...(1.4.6)}$$

where $\lambda > 0$ , $f(z)$ is an analytic function in a simply connected region of the z -plane containing the origin, and the multiplicity of $(z-\zeta)^{1-\lambda}$ is removed by requiring log $(z-\zeta)$ to be real when $z-\zeta > 0$.

The fractional derivative of order $\lambda$ is defined, for a function $f(z)$, by

$$D_{z}^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad \text{...(1.4.7)}$$

where $0 \leq \lambda < 1, f(z)$ is an analytic function in a simply connected region of the z - plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in (1.4.6) above.

Under the hypotheses of (1.4.7), the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \qquad \text{...(1.4.8)}$$

Where $0 \le \lambda < 1$, and $n \in N_0 = N \cup \{0\}$.

**(ii) Saigo operator**

Saigo [84] studied the generalization of Riemann-Liouville operator involving the Gauss hypergeometric function, for the details we refer to [100, pp.283-306]. These generalized operators are defined and represented as follows:

Let $v > 0$ and $\mu, \rho \in C$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional integral operator $I_0^{v,\mu,\rho}$ of a function $f(x)$ is defined by

$$\left(I_{0+}^{v,\mu,\rho} f\right)(x) = \frac{x^{-v-\mu}}{\Gamma(v)} \int_0^{\infty} (x-t)^{v-1} \, {}_2F_1\left(v+\mu, -\rho; v; 1-\frac{t}{x}\right) f(t)\, dt \qquad \text{...(1.4.9)}$$

$$\left(I_{-}^{v,\mu,\rho} f\right)(x) = \frac{1}{\Gamma(v)} \int_x^{\infty} (t-x)^{v-1} t^{-v-\mu} \, {}_2F_1\left(v+\mu, -\rho; v; 1-\frac{x}{t}\right) f(t)\, dt \qquad \text{...(1.4.10)}$$

**(iii) Erdelyi-Kober fractional integral and derivative operators**

Erdelyi-Kober fractional integral and derivative operators are defined in Kiryakova [51]. It is defined as follows:

$$\left(I_{\rho,v}^{+} f\right)(x) = \frac{x^{-v-\rho}}{\Gamma(v)} \int_0^{x} (x-t)^{v-1} t^{\rho} f(t)\, dt \qquad \text{…(1.4.11)}$$

$$\left(K_{\rho,v}^{-} f\right)(x) = \frac{x^{\rho}}{\Gamma(v)} \int_x^{\infty} (t-x)^{v-1} t^{-v-\rho} f(t)\, dt \qquad \text{…(1.4.12)}$$

$$\left(D_{\rho,v}^{+} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_{0+}^{-v+m,-v,v+\rho-m} f\right)(x) \qquad \text{…(1.4.13)}$$

$$\left(D_{\rho,v}^{-} f\right)(x) = \left(-\frac{d}{dx}\right)^m \left(I_{-}^{-v+m,-v,v+\rho} f\right)(x) \qquad \text{…(1.4.14)}$$

Where $\rho, v \in C,\ \operatorname{Re}(v) > 0,\ m = [\operatorname{Re}(v)] + 1 \ \textit{and}\ x \in R^{+}$.

**(iv) Marichev-Saigo-Maeda operators**

Marichev [64] and Saigo and Maeda [83] studied the generalized fractional calculus operators with the Appell function $F_3$ in their kernel. These operators are known as Marichev-Saigo-Maeda operators and defined as follows:

Let $\nu,\nu',\mu,\mu',\eta\in\mathbb{C}$ with $\mathrm{Re}(\eta)>0$ $x\in R^{+}$ , then the left and right fractional integral operators are defined as follows [83]:

$$\left(I_{0+}^{\nu,\nu',\mu,\mu',\eta}f\right)(x)=\frac{x^{-\nu}}{\Gamma(\eta)}\int_{x}^{\infty}(x-t)^{\eta-1}t^{-\nu'}F_3\left(\nu,\nu',\mu,\mu';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\,dt \quad \ldots(1.4.15)$$

$$\left(I_{-}^{\nu,\nu',\mu,\mu',\eta}f\right)(x)=\frac{x^{-\nu'}}{\Gamma(\eta)}\int_{x}^{\infty}(t-x)^{\eta-1}t^{-\nu}F_3\left(\nu,\nu',\mu,\mu';\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,dt \quad \ldots(1.4.16)$$

where $F_3$ is defined as follows [96]:

$$F_3(\nu,\nu',\mu,\mu';\eta;x,t)=\sum_{m,n=0}^{\infty}\frac{(\nu)_m(\nu')_n(\mu)_m(\mu')_n}{(\eta)_{m+n}}\frac{x^m}{m!}\frac{t^n}{n!} \qquad \left(\max\{|x|,|t|\}<1\right)$$

$$\ldots(1.4.17)$$

And for $\nu,\nu',\mu,\mu',\eta\in\mathbb{C}$ with $\mathrm{Re}(\eta)>0$ , $x\in R^{+}$, the left and right Marichev-Saigo-Maeda fractional differential operators are defined in the following manner [83]:

$$\left(D_{0+}^{\nu,\nu',\mu,\mu',\eta}f\right)(x)=\left(I_{0+}^{-\nu',-\nu,-\mu',-\mu,-\eta}f\right)(x)$$

$$=\left(\frac{d}{dx}\right)^{m}\left(I_{0+}^{-\nu',-\nu,-\mu'+m,-\mu,-\eta+m}f\right)(x) \quad \ldots(1.4.18)$$

and

$$\left(D_{-}^{\nu,\nu',\mu,\mu',\eta}f\right)(x)=\left(I_{-}^{-\nu',-\nu,-\mu',-\mu,-\eta}f\right)(x)$$

$$=\left(-\frac{d}{dx}\right)^{m}\left(I_{-}^{-\nu',-\nu,-\mu',-\mu+m,-\eta+m}f\right)(x) \quad \ldots(1.4.19)$$

where $\mathrm{m}=\left[\mathrm{Re}(\eta)\right]+1$and $\left[\mathrm{Re}(\eta)\right]$denotes the integer part of $\mathrm{Re}(\eta)$.

These operators are reduced to be the following Saigo fractional derivative operators [84]:

$$\left(D_{0+}^{\nu+\mu,0,-\rho,0,\nu}f\right)(x)=\left(D_{0+}^{\nu,\mu,\rho}f\right)(x) \quad , \qquad (\rho\in C) \quad \ldots(1.4.20)$$

and

$$\left(D_{-}^{\nu+\mu,0,-\rho,0,\nu} f\right)(x)=\left(D_{-}^{\nu,\mu,\rho} f\right)(x) \quad , \qquad (\rho \in C) \qquad \ldots(1.4.21)$$

**(v) Caputo-type fractional derivative operator**

Recently Rao et.al [82] introduced and studied the Caputo-type fractional derivative that have the Gauss hypergeometric function in the kernel these fractional derivatives are defined in the following manner.

Let $\mu,\nu,\alpha \in C$ with $\operatorname{Re}(\nu)>0, x \in R^{+}$ then the left and right Caputo-type fractional differential operators are defined as follows.

$$\left({}^{C}D_{0+}^{\mu,\nu,\alpha} f\right)(x)=\left(I_{0+}^{-\nu+[\operatorname{Re}(\nu)]+1,-\mu-[\operatorname{Re}(\nu)]-1,\nu+\alpha-[\operatorname{Re}(\nu)]-1} f^{([\operatorname{Re}(\nu)]+1)}\right)(x) \qquad \ldots(1.4.22)$$

and

$$\left({}^{C}D_{-}^{\mu,\nu,\alpha} f\right)(x)=(-1)^{([\operatorname{Re}(\nu)]+1)}\left(I_{0+}^{-\nu+[\operatorname{Re}(\nu)]+1,-\mu-[\operatorname{Re}(\nu)]-1,\nu+\alpha} f^{[\operatorname{Re}(\nu)]+1}\right)(x) \qquad \ldots(1.4.23)$$

**(vi) Caputo-type Marichev-Saigo-Maeda fractional derivative**

Recently Kataria and Vellaisamy [46] studied The Caputo-type Marichev-Saigo-Maeda fractional differential operators in the following form:

Let $\nu,\nu',\mu,\mu',\xi \in C$ with $\operatorname{Re}(\xi)>0, x \in R^{+}$ then the left and right Caputo-type Marichev-Saigo-Maeda fractional differential operators are defined as follows,

$$\left({}^{C}D_{0+}^{\nu,\nu',\mu,\mu',\xi} f\right)(x)=\left(I_{0+}^{-\nu',-\nu,-\mu'+[\operatorname{Re}(\nu)]+1,-\mu-\xi+[\operatorname{Re}(\nu)]+1} f^{([\operatorname{Re}(\nu)]+1)}\right)(x) \qquad \ldots(1.4.24)$$

and

$$\left({}^{C}D_{-}^{\nu,\nu',\mu,\mu',\xi} f\right)(x)=(-1)^{[\operatorname{Re}(\nu)+1]}\left(I_{-}^{-\nu',-\nu,-\mu',-\mu+\operatorname{Re}(\nu)+1,-\xi+[\operatorname{Re}(\nu)]-1,\nu+\alpha} f^{[\operatorname{Re}(\nu)]+1}\right)(x) \qquad \ldots(1.4.25)$$

**(vii) Liouville-Caputo fractional derivative operator**

Liouville [62] and Caputo [6] defined a fractional derivative of order $\mu>0$ of a function $f(t)$ as follows.

$$\frac{d^{\mu}}{dt^{\mu}}\{f(x)\}=\left({}^{LC}D_{0+}^{\mu} f\right)(x)=\begin{cases} f^{(n)}(x) & , \quad \mu=n \in N_{0} \\ \dfrac{1}{\Gamma(n-\mu)} \displaystyle\int_{0}^{x} \frac{f^{(n)}(t)}{(x-t)^{\mu-n+1}} dt, & (n-1<\operatorname{Re}(\mu)<n; n \in N) \end{cases}$$

$$\ldots(1.4.26)$$

known as Liouville-Caputo fractional derivative.

where $n=\begin{cases}[\mathrm{Re}(\mu)]+1 & (\mu \neq N_0)\\ \mu & (\mu \in N_0)\end{cases}$ …(1.4.27)

**(viii) Hilfer fractional derivative operators**

The Two-parameter family of fractional derivatives of order is $(0<\mu<1)$ and type $v(0\le v\le 1)$ was introduced and studied recently by Hilfer in the following manner (see [37],[38] and [39]; see also [36] and [112]):

$$\left({}^{H}D_{a\pm}^{\mu,v} f\right)(x)=\pm\left({}^{RL}I_{a\pm}^{v(1-\mu)}\frac{d}{dx}\left({}^{RL}I_{a\pm}^{(1-v)(1-\mu)} f\right)\right)(x) \quad \dots(1.4.28)$$

Here, the right sided and left sided Hilfer fractional derivative of order $\mu$ and type $v$ with respect to $x$ are defined, in terms of the Riemann-Liouville fractional integral operator defined in (1.4.2) and (1.4.3).

In particular at $v=0$ *and* $v=1$ Hilfer fractional derivative operator defined in (1.4.28) reduce to Riemann-Liouville fractional derivative and familiar Caputo fractional derivative operators respectively.

Today fractional calculus models find applications in physical, biological, engineering, biomedical and earth sciences. Most of the problems discussed involves relaxation and diffusion models. Thus, it gives rise to the generalization of initial value problems involving ordinary differential equations to generalized fractional order differential equations and Cauchy problems involving partial differential equations to fractional reaction and fractional diffusion equations. Apart from the above work fractional calculus is studied by Oldham and Spanier in [73]. One of the most recent works on the subject of fractional calculus is the book of Podlubny [76], some of the latest works especially on fractional models of anomalous kinetics of complex processes are the volumes edited by Carpinteri and Mainardi [8] and by Hilfer [37] and the book of Zaslavsky [116]. Indeed, in the mean time, numerous other works have also appeared. These include the remarkable comprehensive encyclopedic type monograph by Samko, Kilbas and Marichev [87], the book devoted substantially to fractional differential equations by Miller and Ross [65], Prof. V. Kiaryakova also published a book [51] and very recently a monograph by Kilbas, Srivastava and Trujillo [48] presents the upto date work on fractional differential equations.

## 1.5 INTEGRAL TRANSFORMS

Let $f(t)$ be a function of $t$ defined in an interval $(a,b)$. Let $k(p,t)$ be a known function of two parameters $p$ and $t$; where $p$ is a parameter (may be real and complex) independent of $t$, then the integral equation

$$T\left[f(t);p\right]=\int_a^b k(p,t)\,f(t)\,dt \qquad ...(1.5.1)$$

is called the integral transform of the function $f(t)$ provided (1.5.1) is convergent. The function $f(p,t)$ appearing in integrand is called kernel of the Transform.

If we choose

$$k(p,t)=\begin{cases} 0 & ,t<0 \\ e^{-pt} & ,t\geq 0 \end{cases} \qquad ...(1.5.2)$$

Then the transform (1.5.1) leads us to the Laplace transform.

**(a) Laplace Transform**

Let $f(t)$ be a function of $t$ defined for all the positive values of $t$ then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$, is defined by

$$L\{f(t):p\}=\int_0^\infty e^{-pt}f(t)\,dt \qquad ...(1.5.3)$$

provided that the integral exits. The parameter $p$ is a real or complex number. In general, $p$ is taken to be positive real number.

$L\{f(t)\}$ being a function of $p$ is briefly written as $\overline{f}(p)$ i.e.

$$L\{f(t):p\}=\overline{f}(p)=\int_0^\infty e^{-pt}f(t)\,dt \qquad ...(1.5.4)$$

**(b) Inverse Laplace Transform**

If the Laplace transform of a function $f(t)$ is $\overline{f}(p)$, i.e. $L\{f(t):p\}=\overline{f}(p)$, then $f(t)$ is called an inverse Laplace transform of the function $\overline{f}(p)$ and is written as

$$f(t)=L^{-1}\{\overline{f}(p):t\} \qquad ...(1.5.5)$$

where $L^{-1}$ is called the inverse Laplace transformation operator.

**(c) Convolution**

Let $f(t)$ and $g(t)$ be the two functions such that

(i) They are piecewise continuous in $0 \le t \le T$

(ii) They are of exponential order $\alpha$ .

(iii) $L^{-1}\left\{\bar{f}(p):t\right\} = f(t);\ L^{-1}\left\{\bar{g}(p):t\right\} = g(t)$

then the convolution of the two functions $f(t)$ and $g(t)$ denoted by $f * g$ is defined by

$$f * g = L^{-1}\left\{\bar{f}(p).\bar{g}(p)\right\} = \int_0^t f(u)\,g(t-u)\,du \qquad \text{...(1.5.6)}$$

More detailed information may be found in the books by Ditkin and Prudnikov [14] and Doetsch [15], [16].

**(d) Laplace transform of derivatives**

The Laplace transform for ordinary derivative $f^{(n)}(t)$ of order $n \in N_0$ is given as follows:

$$L\left\{f^{(n)}(t):s\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(t)\Big|_{t=0} \qquad (n \in N_0) \qquad \text{...(1.5.7)}$$

Or

$$L\left\{f^{(n)}(t):s\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+) \qquad (n \in N_0) \qquad \text{...(1.5.8)}$$

Laplace transform formula for right sided Liouville-Caputo fractional derivative $\left({}^{LC}D_{0+}^{\mu} f\right)(t)$ is given as follows (see[ 76, p.80, eq. (2.140)] and [ 48, p.98, eq. (2.4.62)])

$$L\left\{\left({}^{LC}D_{0+}^{\mu} f\right)(t):s\right\} = s^{\mu} F(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} \frac{d^k}{dt^k}\left\{f(t)\right\}\Big|_{t=0} \qquad \text{...(1.5.9)}$$

$$= s^{\mu} F(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} f^{(k)}(0+) \qquad \text{...(1.5.10)}$$

$$\left(n-1 < \operatorname{Re}(\mu) \le n;\ n \in N\right)$$

Laplace transform formula for the right sided Hilfer fractional derivative operator ${}^{H}D_{0+}^{\mu,\nu}$ of order $\mu$ and type $\nu$ is given as follows (see [94, p.2651, eq. (1.15)]):

$$L\left\{\left({}^{H}D_{0+}^{\mu,\nu} f\right)(t);s\right\} = s^{\mu} L\left\{f(t);s\right\} - s^{\nu(\mu-1)}\left({}^{RL}I_{0+}^{(1-\nu)(1-\mu)} f\right)(0+) \qquad \text{...(1.5.11)}$$

where $\left({}^{RL}I_{0+}^{(1-\nu)(1-\mu)} f\right)(0+)$ is the Riemann-Liouville integral of order $(1-\nu)(1-\mu)$.

## 1.6 KINETIC EQUATIONS

Initially, Abel (1823), Riemann, and Liouville laid the groundwork for fractional kinetic equations by creating fractional operators and in the twentieth century. Mittag-Leffler, Lévy, Feller, and others linked these operators to physical applications. More recently, significant contributions by Haubold, Mathai, Saxena, Kilbas, and Srivastava have established FKEs as powerful tools in modeling processes such as anomalous diffusion in porous media, viscoelastic relaxation, reaction-diffusion systems, turbulence in plasmas, population dynamics, astrophysical transport, and epidemiological spread.

Consider an arbitrary nuclear reaction process characterized by a time-dependent variable $N(t)$. The rate of change of this quantity can be expressed as the balance between its production rate $p$ and destruction rate $d$ :

$$\frac{dN}{dt} = -d + p \qquad \ldots(1.6.1)$$

In realistic astrophysical systems, however, both the production and destruction rates depend on the current state of the system, i.e., $d = d(N)$ and $p = p(N)$. These rates are not only functions of the instantaneous value $N(t)$ but also depend on the past history of the system, reflecting the inherent memory effects in stellar plasmas.

Thus, the general reaction-type equation can be formally written as (see [34]) :

$$\frac{dN}{dt} = -d(N_t) + p(N_t) \qquad \ldots(1.6.2)$$

Where $N = N(t)$ characterizes the rate of reaction, $d = d(N)$ signifies the rate of destruction , and $p = p(N)$ represents the rate of production and the function $N(t)$ represents the function $N_t(t^*) = N(t - t^*)$, for positive parameter $t^*$ .

For the specific case where we neglect spatial fluctuations and inhomogeneities in $N(t)$, we arrive at the equation :

$$\frac{dN}{dt} = -c_i N_i(t) \qquad \ldots(1.6.3)$$

With the initial condition $N_i(t = 0) = N_0$, which represents the number density of species $i$ at time $t=0$ and $c_i>0$. If we generalize this equation by removing the index and integrating the standard kinetic equation, we obtain :

$$N(t)-N_0=-c\,{}_0D_t^{-1}N(t) \qquad \ldots(1.6.4)$$

Here, ${}_0D_t^{-1}$ represents a special case of the Riemann-Liouville integral operator .

The fractional generalization of the standard kinetic equation is provided by Haubold and Mathai [34] as :

$$N(t)-N_0=-c^{\mu}\,{}_0D_t^{-1}N(t) \qquad \ldots(1.6.5)$$

They further derive the solution for the above equation as follows :

$$N(t)=N_0\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\mu n-1)}(ct)^{\mu n} \qquad \ldots(1.6.6)$$

Additionally, Saxena and Kalla [89] explore a fractional kinetic equation :

$$N(t)-N_0 f(t)=-c^{\mu}\,{}_0D_t^{-1}N(t), \qquad \mathrm{Re}(\mu)>0 \qquad \ldots(1.6.7)$$

In this equation, $N(t)$ represents the number density of a specific species at time $t$, $N_0$ is the initial number density of that species at $t=0$, $c$ is a constant , and $f$ belongs to the function space $L(0,\infty)$ .

## 1.7 GLOSSARY O F SYMBOLS USED

In the thesis, the reference to the work of other authors have been indicated in square brackests i.e. [9]. Thus [5, p.35, eq. (1.3.9)] would mean that the results referred to is given by equation number (1.3.9) on the page 35 of the paper (or the book) at serial number 5 in the list of references. The result of the text of the present work have been specified in parenthesis ( ); for example (4.3.9) refers to equation number (9) of the third section in fourth chapter of the thesis.

The glossary of the symbols used in the thesis is as follows:

| | **Symbol** | **Stands for** |
|---|---|---|
| 1. | $(z)_n$ | $z(z+1)\ldots(z+n-1)$ or $\dfrac{\Gamma(z+n)}{\Gamma(z)}$ |
| 2. | $(z)_0$ | 1 |

| | | |
|---|---|---|
| 3. | $(a_p)$ or $(a_j)_{1,p}$ | $a_1, a_2, ..., a_p$ |
| 4. | $(a_j, \alpha_j)_{1,p}$ | $(a_1, \alpha_1), (a_2, \alpha_2), ..., (a_p, \alpha_p)$ |
| 5. | $\operatorname{Re}(\alpha)$ | Real part of $\alpha$ |
| 6. | $\sum_{k_1,...,k_r=0}^{\infty}$ | $\sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty}$ |
| 7. | $\Gamma(z)$ | Gamma function of $z$ |
| 8. | $B(\alpha, \beta)$ | Beta function |
| 9. | ${}_2F_1(.)$ | Gaussian hypergeometric function |
| 10. | ${}_pF_q[.]$ | Generalized hypergeometric function |
| 11. | ${}_1F_1[.]$ or $\phi(\gamma; \mu; z)$ | Kummer's Confluent hypergeometric function |
| 12. | ${}_p\psi_q[.]$ or ${}_p\psi_q^*[.]$ | Fox-Wright's generalized hypergeometric function |
| 13. | $F_{q;q_1;...;q_r}^{p;p_1;...;p_r}$ or $F[z_1, ..., z_n]$ | Multivariable Kampè-de-Fèriet function |
| 14. | $\binom{n}{r}$ | $\frac{n!}{r!(n-r)!}$ or $\frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$ |
| 15. | $E_\alpha(z)$ | Classical Mittag-Leffler function |
| 16. | $E_{\alpha,\beta}(z)$ | Classical Mittag-Leffler function |
| 17. | $E_{\alpha,\beta}^{\gamma}(z)$ | Generalized three parameter Mittag-Leffler function |
| 18. | $E_{\alpha,\beta}^{\gamma,q}(z)$ | Generalized four parameter Mittag-Leffler function |
| 19. | $E_{(1/\alpha_i),(\beta_i)}^{(m)}(z)$ <br> $E_{1,1}[(1/\alpha_1, \beta_1), ..., (1/\alpha_m, \beta_m); z]$ | Multi-index Mittag-Leffler function |
| 20. | $E_{(\rho_j),\lambda}^{(\gamma_j)}(z_1, ..., z_r)$ | Multivariable Mittag-Leffler function |
| 21. | $E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1, ..., z_r)$ | Multivariable Extended Mittag-Leffler function |

| | | |
|---|---|---|
| 22. | $E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)$ | Multi-index Mittag-Leffler function |
| 23. | $E_{\alpha,\beta,v,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$ | Generalized Multi-index Mittag-Leffler function |
| 24. | $E_{(\rho_j;\lambda;q_j)}^{(\gamma_j;l_j;p_j)}(z_1,...,z_m)$ | Generalized Multivariable Multi-index Mittag-Leffler function |
| 25. | $E_{(\rho_1^{(r)}),...,(\rho_m^{(r)});\beta_1,...,\beta_m}^{(\gamma_r),(l_r)}(z_1,...,z_r)$ | Generalized Multi-index Multivariable Mittag-Leffler function |
| 26. | $(a)_{n,k}$ | Pochhammeer k-symbol |
| 27. | $\Gamma_k(z)$ | k-gamma function |
| 28. | $B_k(x,y)$ | k-Beta function |
| 29. | $B(x,y;p)$ | Extended beta function |
| 30. | $B_k(x,y;p)$ | Extended k-Beta function |
| 31. | $E_{k,l,m}^{\rho}(x)$ | k-Mittag-Leffler function |
| 32. | $E_{k,l,m}^{\rho;c}(x;p)$ | Extended k-Mittag-Leffler function |
| 33. | $E_{k,l,m}^{\rho,\sigma;c}(x;p)$ | Extended k-generalized Mittag-Leffler function |
| 34. | $E_{v,\mu}^{j,k}(z)$ | Mittag-Leffler type function for arbitrary order |
| 35. | $E_{\alpha,\beta}^{\rho,c;j,\gamma}(z;p)$ | Extended Mittag-Leffler type function for arbitrary order |
| 36. | $\in_{\alpha,\beta}(\phi;z)$ | Taylor-Maclaurin series |
| 37. | $E_{\alpha,\beta}^{(k)}(s;z)$ | Barnes Class of functions |
| 38. | $L\{f(t)\}$ | Laplace transform of function |
| 39. | $L^{-1}\{\overline{f}(p)\}$ | Inverse Laplace transform operator |
| 40. | $D^n$ of $\dfrac{d^n}{dz^n}$ | $n^{th}$ derivative operator |
| 41. | $I_{a+}^{\alpha}$ | Left sided Riemann-Liouville fractional integral operator |

| | | |
|---|---|---|
| 42. | $I_{b-}^{\alpha}$ | Right sided Riemann-Liouville fractional integral operator |
| 43. | $D_{a+}^{\alpha}$ | Left sided Riemann-Liouville fractional derivative operator |
| 44. | $D_{b-}^{\alpha}$ | Right sided Riemann-Liouville fractional derivative operator |
| 45. | $D_{z}^{-\lambda}$ | Fractional integral (operator) of order $\lambda$ |
| 46. | $D_{z}^{\lambda}$ | Fractional derivative (operator) of order $\lambda$ |
| 47. | $I_{0+}^{\nu,\mu,\rho}$ | Saigo Left fractional integral operator |
| 48. | $I_{-}^{\nu,\mu,\rho}$ | Saigo Right fractional integral operator |
| 49. | $I_{\rho,\nu}^{+}$ | Erdelyi-Kober Left fractional integral operator |
| 50. | $K_{\rho,\nu}^{-}$ | Erdelyi-Kober Right fractional integral operator |
| 51. | $D_{\rho,\nu}^{+}$ | Erdelyi-Kober Left fractional Derivative operator |
| 52. | $D_{\rho,\nu}^{-}$ | Erdelyi-Kober Right fractional Derivative operator |
| 53. | $I_{0+}^{\nu,\nu',\mu,\mu',\eta}$ | Marichev-Saigo-Maeda Left fractional integral operator |
| 54. | $I_{-}^{\nu,\nu',\mu,\mu',\eta}$ | Marichev-Saigo-Maeda Right fractional integral operator |
| 55. | $D_{0+}^{\nu,\nu',\mu,\mu',\eta}$ | Marichev-Saigo-Maeda Left fractional differential operator |
| 56. | $D_{-}^{\nu,\nu',\mu,\mu',\eta}$ | Marichev-Saigo-Maeda Right fractional differential operator |
| 57. | ${}^{C}D_{0+}^{\mu,\nu,\alpha}$ | Caputo-type fractional Left fractional differential operator |
| 58. | ${}^{C}D_{-}^{\mu,\nu,\alpha}$ | Caputo-type fractional Right fractional |

| | | |
|---|---|---|
| | | differential operator |
| 59. | ${}^{C}D_{0+}^{\nu,\nu',\mu,\mu',\xi}$ | Caputo-type Marichev-Saigo-Maeda left fractional derivative operator |
| 60. | ${}^{C}D_{0-}^{\nu,\nu',\mu,\mu',\xi}$ | Caputo-type Marichev-Saigo-Maeda Right fractional derivative operator |
| 61. | ${}^{LC}D_{0+}^{\mu}$ | Liouville-Caputo fractional derivative operator |
| 62. | ${}^{H}D_{a\pm}^{\mu,\mathrm{v}}$ | Hilfer fractional derivative operator |
| 63. | $\zeta(a)$ | Riemann Zeta function |
| 64. | $\zeta(s,a)$ | Lerch Zeta function |
| 65. | $\phi(z,s,a)$ or $\Phi(z,s,a)$ | Hurwitz-Lerch Zeta function |
| 66. | $\phi_a^{*}(z,s,a)$ | Goyal-Ladda's Hurwitz-Lerch Zeta function |
| 67. | $\phi_{\alpha,\beta,\gamma}(z,s,a)$ | Garg's Hurwitz-Lerch Zeta function |
| 68. | $\phi_{\lambda,\mu}^{(\rho,\sigma)}(z,s,a)$ | Lin-Srivastava Hurwitz-Lerch Zeta function |
| 69. | $\phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(z,s,a)$ | Extended Hurwitz-Lerch Zeta function |
| 70. | $\phi_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_q}^{\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q}(z,s,a)$ | Generalized Hurwitz-Lerch Zeta function |
| 71. | $\theta_1^{\lambda}(x,\alpha,a,b)$ | A class of generalized Hurwitz-Lerch Zeta function of one variable |
| 72. | $\theta_{(p_i)}^{\lambda}(x_1,\ldots,x_r;\alpha,a,b)$ | A class of generalized Hurwitz-Lerch Zeta function of several variable |
| 73. | $\phi_{\lambda,\mu,\nu}^{(a,\rho,\sigma,k)}(z,s,a,b)$ | A class of Extended Hurwitz-Lerch Zeta function |
| 74. | $\phi_{\lambda_j,\mu_k,\beta}^{(a,\rho_j,\sigma_k)}(z,s,a,b)$ | A general class of Hurwitz-Lerch Zeta function |
| 75. | $\phi_{\lambda_j^{(i)},\mu_k^{(i)},\beta_i}^{\left(a,\rho_j^{(i)},\sigma_k^{(i)}\right)}(z_1,\ldots,z_n;s,a,b)$ | A class of multivariable Generalized Extended Hurwitz-Lerch Zeta function |

## 1.8 ABSTRACT OF THESIS

The abstract of rest four chapters is as follows.

**Chapter-2**

In the chapter a general function for single and multivariable cases which unifies the Hurwitz-Lerch Zeta as well as Mittag-Leffler functions are introduced. The integral representations and generating functions for these functions, along with a mild parameter extension are established. The chapter divides into two sections.

In section-A, the single-variable general function $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)$ is studied. The integral representation and four generating functions for this general function are established as main result of this section. These results are further extended in view of parameter. For illustration we present here the following result among the main results established in section-A:

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0$ ; $\alpha,\beta,\mu,\lambda\in C$; $a,\sigma\in C/Z_0^-$ ; $\delta,\nu,\xi\in R^+$; $\xi-\delta-\nu\geq-1$

with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+l,a)t^l=\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a-t) \qquad \ldots(1.8.1)$$

provided that both the side of (1.8.1) exists.

Its mild extension is given as follows:

For all $(i=1,2,\ldots\ldots.\mathrm{p}; j=1,2,\ldots\ldots.\mathrm{q})$, $p,q\in N_0$; $s,z,\mu_i,\alpha,\beta\in C$; $a,\sigma_j\in C/Z_0^-$ ; $\delta_i,\rho_j\in R^+$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s+l,a)t^l=\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a-t) \qquad \ldots(1.8.2)$$

provided that both the side of (1.8.2) exists.

As application of integral representations and generating functions established in section-A certain known results for Hurwitz-Lerch Zeta function $\phi(z,\mathrm{s},a)$ are shown to be derived.

In Section-B, the multivariable analogue of general function studied in section-A is introduced. Here we have established the integral representation of this multivariable general function and then certain generating relations associated with this general function are also established. These results are further extended for its mild extension in the section. For illustration we state one of these results here:

For all $i = 1\ldots \mathrm{r}$; $\delta_i \in R^+,\ \mu_i, \alpha_i, \mathrm{s}, \mathrm{z} \in C$; $min\left\{\mathrm{Re}(a), \mathrm{Re(s)}\right\} > 0$; $\mathrm{Re}(p_i) > 0$;

$\mathrm{Re}(\alpha_i) > 0$; and $\max\limits_{1 \le i \le r}(|z_i|) < 1$,

we have the following generating function

$$\sum_{i=0}^{\infty}\binom{s+2l-1}{2l} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,\ldots,z_r; s+2l, a\right)t^{2l}$$

$$= \frac{1}{2}\left[\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,\ldots,z_r; s, a+t\right) + \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,\ldots,z_r; s, a-t\right)\right] \qquad \ldots(1.8.3)$$

provided that both the side of (1.8.3) exists.

Its mild extension as follows:

For all $\left(i = 1\ldots\mathrm{r};\ j = 1\ldots p_i;\ \mathrm{k} = 1\ldots\mathrm{q}_i\right)$, $p_i, \mathrm{q}_i \in N_0$; $\delta_j^{(i)}, \rho_k^{(i)} \in R^+$; $a, \sigma_k^{(i)} \in C/Z_0^-$

$\gamma_i, s, \mathrm{z}, \mu_j^{(i)}, \alpha_i, \beta \in C$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+2l-1}{2l} \in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,\ldots,z_r; s+2l, a\right)t^{2l}$$

$$= \in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,\ldots,z_r; s, a+t\right) + \in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,\ldots,z_r; s, a-t\right)$$

...(1.8.4)

provided that both the side of (1.8.4) exists.

Certain known and new results associated with multivariable Hurwitz-Lerch Zeta function are shown to be derived as applications of main results established in this section-B.

**Chapter-3**

In this chapter we have investigated integral representations, fractional derivatives, and integral expressions for the general class of functions considered in chapter-2 for one and several variables. It is organized into two section A and B.

In Section-A, four integral representations and one fractional derivative are derived for the single-variable general function, with mild extension.

We have evaluated certain type of integral representation involving the Mittag-Leffler function which gives the general function as resultant and one result of fractional derivative for this general function and these results further extends for its mild extension. We give the following result for citation:

For $w, \lambda, \mu \in C$ ; $\sigma \in C/Z_0^-$ ; $p \in N_0$ ; $\min\{\mathrm{Re}(a), \mathrm{Re}(s)\} > 0$;

$\delta, \nu \in R^+; \rho - \delta - \nu \geq -1$ with equality when $|z| < \xi^\rho = \delta^{-\delta}\nu^{-\nu}\rho^\rho$,

we have the following integral representation

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a) = \frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\int_0^\infty t^{s-1}e^{-(a+kp)t}\left(1-e^{-pt}\right)^w E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)dt \quad \ldots(1.8.5)$$

provided that each side of (1.8.5) exists and $E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittag-Leffler function. As applications of integral representation and fractional derivative established in section-A, some known integral representation and fractional derivatives for simpler function related to the Hurwitz -Lerch Zeta functions are obtained.

Section-B presents the study for the multivariable analogue and its mild extension, one integral representation and three series integral expressions are established for this multivariable general function in term of multivariable Mittag-Leffler function. The following integral representation is present here for illustration:

For all $i = 1\ldots r$; $\delta_i \in R^+$ $\mu_i, \alpha_i, s, z \in C$; $min\{\mathrm{Re}(a), \mathrm{Re}(s)\} > 0$; $\mathrm{Re}(p_i) > 0$;

$\mathrm{Re}(\alpha_i) > 0$; and $\max\limits_{1\leq i\leq r}|z_i| < 1$,

we have the following integral expression

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}(z_1,\ldots,z_r;s,a) = \frac{1}{\Gamma s}\sum_{l=0}^{\infty}\frac{(\gamma)_l}{l!}\int_0^\infty t^{s-1}e^{-(a+lp)t}\left(1-e^{-pt}\right)^\gamma E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1e^{-p_1t},\ldots,z_re^{-p_rt}\right)dt$$

$$\ldots(1.8.6)$$

provided that each the side of (1.8.6) exists and $E_{(\alpha_r),\beta}^{(\mu_r)(\delta_r)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function.

Some known and new results are shown to be obtained for the Hurwitz-Lerch Zeta function as application of our main results.

General function studied in this section unifies all the function related to multivariable Hurwitz-Lerch Zeta function as well as the Mittag-Leffler function. therefore, certain known and new for the multivariable Hurwitz-Lerch Zeta function and related simpler function are derived as application of our main results.

**Chapter-4**

In this chapter we are obtained the fractional integral and derivative formulae for the extended k-generalized Mittag-Leffler function $E_{k,l,m}^{\rho,\sigma;c}\left(x;p\right)$ and the extended k-generalized Mittag-Leffler type function of arbitrary order $E_{k,l,m}^{\rho,\sigma;c;j,\gamma}\left(x;p\right)$ using Marichev-Saigo-Maeda integral and differential operators. Some formula using Caputo-type Marichev-Saigo-Maeda fractional derivative operator for both the extended k-generalized Mittag-Leffler function and the extended k-generalized Mittag-Leffler type function are obtained. This chapter includes section-A and B.

In section-A, six formulas as images under Marichev-Saigo-Maeda operators: two for fractional integration, two for fractional differentiation, and two for Caputo-type Marichev-Saigo-Maeda fractional derivatives are obtained for the extended k-generalized Mittag-Leffler function.

First two formulae deal with the Marichev-Saigo-Maeda left and right fractional integral operators $\left(I_{0+}^{v,v',\mu,\mu',\eta}f\right)(x)$ and $\left(I_{-}^{v,v',\mu,\mu',\eta}f\right)(x)$. Next two formulae associated to the Marichev-Saigo-Maeda left and right fractional derivative operators $\left(D_{0+}^{v,v',\mu,\mu',\eta}f\right)(x)$ and $\left(D_{-}^{v,v',\mu,\mu',\eta}f\right)(x)$. The rest two formulae associated to the Caputo-type left and right Marichev-Saigo-Maeda fractional derivatives operators $\left({}^{C}D_{0+}^{v,v',\mu,\mu',\xi}f\right)(x)$ and $\left({}^{C}D_{-}^{v,v',\mu,\mu',\xi}f\right)(x)$ respectively. We give the following formula among the main results established in section-A for citation:

For $v, v', \mu, \mu', \eta, \lambda, t, l, m, \mathrm{c} \in \mathrm{C}$ with

$\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(v+v'+\mu-\eta), \mathrm{Re}(v'-\mu')\} > 0$,

$\mathrm{Re}(\eta) > 0, \mathrm{Re}(c) > 0, k > 0, \mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0$, $\sigma \in (0,1) \cup N; \mathrm{p} \in R_0^+$,

we have the following Marichev-Saigo-Maeda left fractional integral,

$$\left(I_{0+}^{v,v',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}(t;p)\right]\right)(x) = x^{-v-v'+\eta+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}$$

$$\frac{\Gamma(\lambda+n)\Gamma(-v'+\mu'+\lambda+n)\Gamma(-v-v'-\mu+\eta+\lambda+n)}{\Gamma(\lambda+\mu'+\mathrm{n})\Gamma(-v-v'+\eta+\lambda+n)\Gamma(-v'-\mu+\eta+\lambda+n)}\frac{x^n}{n!} \qquad \ldots(1.8.7)$$

As applications of the main theorems established in this section, the formulae for Riemann-Liouville, Erdelyi-Kober fractional integral and derivative, Saigo and Caputo fractional integral and fractional derivative operators of extended k-generalized Mittag-Leffler function are shown to be derived and certain known and new results for simpler functions are also shown to be obtained here.

In section-B we have considered the extension of k-generalized Mittag-Leffler type function studied by Bin-Saad (2025), then six formulas for Marichev-Saigo-Maeda (MSM) fractional integral and derivative operators, along with Caputo-type MSM fractional derivatives involving this extended k-generalized Mittag-Leffler type function are obtained. The following fractional farmula is presented here for illustration:

For $v, v', \mu, \mu', \xi, \lambda, l, m, \rho, \mathrm{c} \in \mathrm{C}$ with

$\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(-v-v'-\mu'+\xi), \mathrm{Re}(\mu-v)\}$,

$\mathrm{Re}(\xi) > 0, \mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$ and $k > 0$, $\mathrm{p} \in R_0^+$,

$\sigma \in (0,1) \cup N$ and $j \geq 1, \gamma \geq 0$, we have the left Marichev-Saigo-Maeda left fractional derivative.

$$\left(D_{0+}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}(\mathrm{t};p)\right]\right)(x) = x^{\lambda+v+v'-\xi+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{(x^j)^n}{n!}\frac{\Gamma(\lambda+\gamma+nj)\Gamma(v-\mu+\lambda+\gamma+nj)\Gamma(v+v'+\mu'-\xi+\lambda+\gamma+nj)}{\Gamma(-\mu+\lambda+\gamma+nj)\Gamma(v+v'-\xi+\lambda+\gamma+nj)\Gamma(v+\mu'-\xi+\lambda+\gamma+nj)}$$

$\ldots(1.8.8)$

As applications, the main theorems established in this section yield the fractional integral and differential formulas involving Riemann-Liouville, Erdélyi-Kober and Saigo fractional operators. Since this extended k-generalized Mittag-Leffler-type function reduces to the four-parameter Mittag-Leffler function, therefore new and known fractional differentiation and integration formulas for the four-parameter and simpler Mittag-Leffler functions are shown to be derived.

**Chapter-5**

A general family of hybrid-type fractional-order kinetic equations for Liouville-Caputo fractional derivative ${}^{LC}D_{0+}^{\sigma}$ and Hilfer fractional derivative operators involving multivariable general functions associated to multivariable Mittag-Leffler and Hurwitz-Lerch Zeta functions are considered. These fractional order kinetic equations are solved using the Laplace transform method. This chapter is divided into two sections: Section-A and B.

In section-A, three theorems on general hybrid-type fractional-order kinetic equations with Liouville-Caputo fractional derivatives involving the multivariable general functions and also associated to multivariable Mittag-Leffler functions and multivariable Hurwitz-Lerch Zeta functions are given. The solution of above kinetic equations are obtained here. For illustration we state the following theorem:

Under the parametric constraints $c,\gamma,v,\rho \in R^{+}$ and $0<\sigma<1$, let the general function $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,\ldots,z_r;s,a\right)$, exists then the solution of the following generalized fractional order kinetic equation

$$N(t)-N_0 t^{\gamma-1}\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1 t^{v_1},\ldots,z_r\, t^{v_r};k,a\right)=-c^{\rho}\left({}^{LC}D_{0+}^{\sigma}N\right)(t) \qquad \ldots(1.8.9)$$

with initial condition $N(0+)=N(t)\big|_{t=0}=\chi_0$ $\qquad \ldots(1.8.10)$

is given by

$$N(t)=N_0 t^{\gamma-1}\sum_{q=0}^{\infty}(-1)^q\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q+1}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}p_i n_i\right)^{k}\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}$$

$$\frac{1}{\Gamma\left(\sum_{i=1}^{r} v_i n_i + \sigma(q+1) + \gamma\right)} \prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_i n_i}\left(z_i t^{v_i}\right)^{n_i}}{n_i!}\right] + \chi_0 \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma\sigma q+1}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^q$$

$$(t>0) \qquad \ldots(1.8.11)$$

provided that the right hand side of the solution (1.8.11) exists.

As applications, results obtained in section-A are capable to yielding solutions of a significantly large number of known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving Hurwitz-Lerch Zeta function of one and several variables. Certain known results of our findings are shown to be obtained.

In section-B, three theorems on general hybrid fractional-order kinetic equations under Hilfer fractional derivatives for multivariable general, Mittag-Leffler, and Hurwitz-Lerch Zeta functions are obtained. The following theorem is given here for illustration:

Let each of the parametric constraints $\gamma, \rho, c, v_i \in R^+ \ (i = 1 \ldots r)$, $0 < \sigma < 1$ and $0 \le \omega \le 1$ hold true. Suppose also that the multivariable general function $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,\ldots,\mathrm{z}_r;k,a\right)$ exists. If we consider

$$\chi_0(\sigma,\omega) = \left({}^{RL}I_{0+}^{(1-\omega)(1-\sigma)}N\right)(0+) \qquad \ldots(1.8.12)$$

then the solution of the following general hybrid-type family of the fractional-order kinetic equations.

$$N(t) - N_0 t^{\gamma-1} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(\mathrm{z}_1 t^{v_1},\ldots,\mathrm{z}_r\, t^{v_r};k,a\right) = -c^{\rho}\left({}^{H}D_{0+}^{\sigma,\omega}N\right)(t) \qquad \ldots(1.8.13)$$

is given by

$$N(t) = N_0 t^{\gamma-1} \sum_{q=0}^{\infty}(-1)^q \left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q+1} \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\Gamma\left(\sum_{i=1}^{r} v_i n_i + \gamma\right)}{\Gamma\left(\beta + \sum_{i=1}^{r}\alpha_i n_i\right)\left(a + \sum_{i=1}^{r} p_i n_i\right)^k}$$

$$\frac{1}{\Gamma\left(\sum_{i=1}^{r} v_i n_i + \gamma + \sigma(q+1)\right)} \prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_i n_i}\left(z_i t^{v_i}\right)^{n_i}}{n_i!}\right]$$

$$+\chi_o(\sigma,\omega)t^{\sigma+\omega(1-\sigma)-1}\sum_{q=0}^{\infty}\frac{(-1)^q}{\Gamma\left[\sigma(q+1)+\omega(1-\sigma)\right]}\left(\frac{t^\sigma}{c^\rho}\right)^q \ (t>0)$$

…(1.8.14)

provided that the right hand side of the solution (1.8.14) exists.

As applications of the theorems established in this section, certain known and new results on fractional order kinetic equations and their solutions involving multivariable general functions and associated functions for Riemann-Liouville fractional derivative ${}^{RL}D_{0+}^{\sigma}$ and Liouville-Caputo fractional derivative operator ${}^{LC}D_{0+}^{\sigma}$ are shown to be derived.

# CHAPTER-2

# GENERATING FUNCTIONS INVOLVING GENERAL SPECIAL FUNCTION RELATED TO HURWITZ-LERCH ZETA FUNCTION AND ITS MULTIVARIABLE GENERALIZATION

# II

The main results of this chapter have been published / accepted / communicated for publication as per details below:

1. Generating Function Involving General Function Related to Hurwitz - Lerch Zeta function.
   Communications on Applied Nonlinear Analysis Vol. 32, No. 8s (2025), 951-962.

2. Multivariable extension of general class of functions associated with the Hurwitz - Lerch Zeta function and certain generating relations.
   (Communicated for publication)

## 2.1 INTRODUCTION

In this chapter, we have defined and studied a general function of single and multivariable which unifies the Hurwitz-Lerch Zeta function and Mittag-Leffler function. The integral representation and certain generating functions involving general functions of single and multivariable are established in this chapter. A mild extension of these general functions in view of parameters is also presented here. Generating functions and integral representation for this extended function are also studied in this paper. This chapter is organized in to two sections A and B. In Section-A, we have established integral representation and certain generating functions for the general function for single variable. In Section-B, we consider the multivariable analogue of this general function. Integral representation and some generating functions are obtained for this multivariable general function.

In Section-A, the integral representation and four generating functions for the general function of one variable and then results are further established for its mild extension. As applications of integral representation and generating functions established in Section-A, certain known results related to Hurwitz-Lerch Zeta function are shown to be derived here.

In Section-B, we have studied a multivariable analogue of general function which unifies the multivariable Hurwitz-Lerch Zeta function and multivariable Mittag-Leffler function. The mild extension for this general function also presented here. We have obtained the integral representation and four generating relations for this multivariable general function and similar results are further established for its mild extension. Many known and new results are shown to be obtained for the Hurwitz-Lerch Zeta function as application of our main results established here.

## 2.2 DEFINITIONS AND PRILIMINARIES

A general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in the literature in the following manner [19]:

$$\phi(z,\mathrm{s},a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \qquad \ldots(2.2.1)$$

($a\in C/Z_0^-$, $\mathrm{s}\in C$ when $|z|<1$ and $\mathrm{Re}(s)>1$ when $|z|=1$)

where C represents the Set of complex numbers and Z represents set of integer numbers and $Z_0^- = Z^- \cup \{0\}$.

The Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in (2.2.1) contains, its special cases, as the Riemann Zeta function $\zeta(s)$ and Hurwitz-Lerch Zeta function $\zeta(s,a)$ defined by (see for details, [19, Chapter 1])

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} = \phi(1,s,1) \quad (\mathrm{Re}(s) > 1) \qquad \ldots(2.2.2)$$

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \phi(1,s,a) \qquad (\mathrm{Re}(s) > 1; a \in C/Z_0^-) \qquad \ldots(2.2.3)$$

A generalization of the Hurwitz-Lerch Zeta function $\phi(z,s,a)$ was studied by Lin and Srivastava [58, p.727, eq. (8)]

$$\phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s} \qquad \ldots(2.2.4)$$

$(\mu \in C; a, \nu \in C/Z_0^-, \rho, \sigma \in R^+; \rho < \sigma$ when $s, z \in C; \rho = \sigma$ and $s \in C$ when $|z| < \delta = \rho^{-\rho}\sigma^{\sigma}; \rho = \sigma$ and $\mathrm{Re}(s - \mu - \nu) > 1$ when $|z| = \delta)$

where $(\lambda)_n$ denotes the general pochhammer symbol which is defined by Rainville [81] in terms of the gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma\lambda} = \begin{cases} 1 & (n = 0; \lambda \in C/\{0\}) \\ \lambda(\lambda+1)\ldots\lambda(\lambda+n-1) & (n = N; \lambda \in C) \end{cases} \qquad \ldots(2.2.5)$$

The function at $\rho = \sigma = 1$ and $\nu = 1$ yields the Hurwitz-Lerch Zeta function defined and studied by Goyal and Laddha [28, p.100, eq. (1.5)]

$$\phi_{\mu}^{*}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s} \qquad \ldots(2.2.6)$$

$(\mu \in C; a \in C/Z_0^-, s \in C$ when $|z| < 1$ and $\mathrm{Re}(s - \mu) > 1$ when $|z| = 1)$.

Further generalization of the functions $\phi(z,s,a)$ and $\phi_{\mu}^{*}(z,s,a)$ was considered by Garg et al. [23] in the following manner [23, p.313, eq. (1.7)]:

$$\phi_{\lambda,\mu,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n\, n!}\frac{z^n}{(a+n)^s} \qquad \ldots(2.2.7)$$

($\lambda,\mu\in C;\ \nu,a\in C/Z_0^-; s\in C$ when $|z|<1$ and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=1$)

A further generalization of a family of Hurwitz-Lerch Zeta function defined and studied by Srivastava et al [101, p.491, eq. (1.20)] in the following form:

$$\phi_{\lambda,\mu;\nu}^{(\rho,\sigma,k)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}\, n!}\frac{z^n}{(n+a)^s} \qquad \ldots(2.2.8)$$

($\lambda,\mu\in C;\nu,a\in C/Z_0^-;\rho,\sigma,\mathrm{k}\in \mathrm{R}^+;k-\rho-\sigma>-1$ when $s,z\in C;k-\rho-\sigma=-1$ and $s\in C$ when $|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}\mathrm{k}^k;\ k-\rho-\sigma=-1$ and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=\delta^*$).

The well known Mittag-Leffler function $E_\alpha(z)$ in defined by following manner [66] (See also [67,113])

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ (\mathrm{z}\in C,\mathrm{Re}(\alpha)>0) \qquad \ldots(2.2.9)$$

This function further generalized by Prabhakar [78] in the following form [See also [109]]:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!} \qquad \ldots(2.2.10)$$

$(\alpha,\beta,\gamma\in C;\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0)$

In view of extension of parameters, Mittag-Leffler function was also studied by Khan and Ahmed [47, p.2, eq. (1.9)]:

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\eta,\mathrm{q}}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}(\delta)_{pn}}\frac{z^n}{\Gamma(\alpha n+\beta)} \qquad \ldots(2.2.11)$$

where $\alpha,\beta,\eta,\delta,\mu,\nu,\rho,\sigma\in C;\ p,q>0$ and $\mathrm{q}\le\mathrm{Re}(\alpha)+p$ and

$\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\delta),\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$ if $\delta=p=1$, it takes the following form

$$E_{\alpha,\beta,\nu,\sigma}^{\mu,\rho,\eta,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(2.2.12)$$

where $\alpha,\beta,\eta,\mu,\nu,\rho,\sigma\in C;\ p,q>0$ and $q\le \mathrm{Re}(\alpha)$ and

$\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$

Wright Introduced and studied the Taylor-Maclaurin Series [114, p.424]:

$$\in_{\alpha,\beta}(\phi;z)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{\Gamma(\alpha n+\beta)} \qquad (\alpha,\beta\in C;\mathrm{Re}(\alpha)>0) \qquad \ldots(2.2.13)$$

Where $\phi(n)$ is a function satisfying suitable conditions.

Motivated by the work of Wright [114], E.W. Barnes [4] considered the asymptotic expansion of functions in the class which is defined as follows:

$$E_{\alpha,\beta}^{(k)}(s;z)=\sum_{n=0}^{\infty}\frac{z^n}{(n+k)^s\Gamma(\alpha n+\beta)} \qquad (\alpha,\beta\in C;\mathrm{Re}(\alpha)>0) \qquad \ldots(2.2.14)$$

for suitable restricted parameters $k$ and $s$.

Recently for suitably restricted sequence $\{\phi(n)\}_{n=0}^{\infty}$, Srivastava [94] introduced and studied a class of function (See also [105],[110]):

$$\in_{\alpha,\beta}(\phi;z,s,a)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{(a+n)^s\Gamma(\alpha n+\beta)} \qquad \ldots(2.2.15)$$

where $(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0)$

For $\phi(n)=\frac{(\mu)_n}{n!}$ the following function recently studied by Kumar [56, p.13, eq. (1.9)]:

$$\in_{\alpha,\beta,\delta}^{\mu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(\delta n+a)^s\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(2.2.16)$$

where $\mathrm{Re}(\mu)>0$; $\mathrm{Re}(\delta)>0$; $\mathrm{Re}(a)>0$; $\mathrm{Re}(s)\ge 0$; $\mathrm{Re}(\alpha)>0$; $\mathrm{Re}(\beta)>0$ and $|z|\le 1$

To give more extension in view of the parameters, we consider here the following more generalized function defined and represented as below.

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}\, z^n}{(\sigma)_{\xi n}(a+n)^s\,\Gamma(\alpha n+\beta)\,n!} \quad \ldots(2.2.17)$$

where $(\alpha,\beta,\mu,\lambda\in C$; $a,\sigma\in C/Z_0^-;\delta,\nu,\xi\in R^+;\xi-\delta-\nu>-1$ when $s,z\in C$;

$\xi-\delta-\nu=-1$ and $s\in C$ when $|z|<\rho^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$; $\xi-\delta-\nu=-1$ and

$\mathrm{Re}(s+\sigma-\mu-\lambda)>1$ when $|z|=\rho^*)$.

The following definitions and facts are required in the sequel.

(i) The Gamma integral is defined in [29, p.884, eq.312(2)] the following form

$$\frac{\Gamma\alpha}{a^{\alpha}}=\int_0^{\infty}x^{\alpha-1}e^{-ax}dx \qquad (\mathrm{Re}(\alpha)>0) \quad \ldots(2.2.18)$$

(ii) The binomial expansions are defined in [96, p.20, eq. (29)] the following form (see also [23])

$$\sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n=(1-x)^{-a} \quad \ldots(2.2.19)$$

$$\sum_{n=0}^{\infty}\frac{(a)_{2n}}{(2n)!}x^{2n}=\frac{1}{2}\left[(1-x)^{-a}+(1+x)^{-a}\right] \quad \ldots(2.2.20)$$

and

$$\sum_{n=0}^{\infty}\frac{(a)_{2n+1}}{(2n+1)!}x^{2n+1}=\frac{1}{2}\left[(1-x)^{-a}-(1+x)^{-a}\right] \quad \ldots(2.2.21)$$

(iii) The generalized hypergeometric function ${}_pF_q$ defined by [96, p.19. eq. (23)] in the following form:

$${}_pF_q\left[\begin{matrix}(a_p); \\ (b_q);\end{matrix}\; z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!}; \qquad (|z|<1) \quad \ldots(2.2.22)$$

## SECTION-A

**2.3** In this section we have obtained the integral representation and four generating functions involving the general function $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)$ defined in (2.2.17). As applications of the main results of this section we explore a number of integral representations and generating functions related to Hurwitz-Lerch Zeta function.

## 2.4 INTEGRAL REPRESENTATION

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \alpha,\beta,\mu,\lambda\in C;\ a,\sigma\in C/Z_0^-\ ;\delta,\nu,\xi\in R^+;\xi-\delta-\nu\geq -1$ with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$; integral representation of the function defined in (2.2.17) $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)$ holds true.

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}\left\{E_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)\right\}dt \qquad \ldots(2.4.1)$$

where $E_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittag-Leffler function defined in (2.2.12).

**Proof of (2.4.1)**

The function defined in (2.2.17) can be written straightly in the following form

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}\ z^n}{(\sigma)_{\xi n}\ n!\Gamma(\alpha n+\beta)}\left\{\frac{\Gamma s}{(a+n)^s}\right\}$$

Now using the following elementary Gamma integral (2.2.18)

$$\frac{\Gamma s}{(a+n)^s}=\int_0^\infty t^{s-1}e^{-(a+n)t}dt$$

and then changing the order of integration and summation, we have

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}\left\{\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\xi n}\Gamma(\alpha n+\beta)}\frac{\left(ze^{-t}\right)^n}{n!}\right\}dt$$

On interpreting the inner series in view of (2.2.12) we at once arrive at the desired result (2.4.1).

Throughout the chapter, we use the notation $\binom{s+l-1}{l}=\frac{(s)_l}{l!}$, where $(s)_l$ is the Pochhammer's notation.

## 2.5 GENERATING FUNCTIONS

The four generating functions involving the general function defined in (2.2.17) are establish here.

**Theorem-1**

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \alpha,\beta,\mu,\lambda\in C;\ a,\sigma\in C/Z_0^-;\ \delta,\nu,\xi\in R^+;\ \xi-\delta-\nu\geq -1$

with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+l,a)t^l=\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a-t) \quad \ldots(2.5.1)$$

provided that both the side of (2.5.1) exists.

**Theorem-2**

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \alpha,\beta,\mu,\lambda\in C;\quad a,\sigma\in C/Z_0^-;\ \delta,\nu,\xi\in R^+;\ \xi-\delta-\nu\geq -1$

with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+2l-1}{2l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+2l,a)t^{2l}=\frac{1}{2}\left[\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a+t)+\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a-t)\right] \quad \ldots(2.5.2)$$

provided that both the side of (2.5.2) exists.

**Theorem-3**

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \alpha,\beta,\mu,\lambda\in C;\quad a,\sigma\in C/Z_0^-;\ \delta,\nu,\xi\in R^+;\ \xi-\delta-\nu\geq -1$

with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$, we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s+2l+1,a\right)t^{2l+1}=\frac{1}{2}\left[\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a-t\right)-\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a+t\right)\right]$$

…(2.5.3)

provided that both the side of (2.5.3) exists.

**Theorem-4**

For $\operatorname{Re}\left(\sum_{i=1}^{p}\theta_i\right)>\operatorname{Re}\left(\sum_{j=1}^{q}\beta_j\right)>0;\quad\left|\frac{t}{a}\right|<1$

we have the following generating relation

$$\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j+l,a\right)\frac{t^l}{l!}$$

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\xi n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\frac{1}{(a+n)^{\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j}}\;{}_pF_q\left[\begin{matrix}(\theta_p);\\(\beta_q);\end{matrix}\frac{t}{a+n}\right]\quad\ldots(2.5.4)$$

where ${}_pF_q$ is the generalized hypergeometric function defined in (2.2.22).

**OUTLINE OF PROOFS**

**Proof of (2.5.1)**

Let left hand side of (2.5.1) is denoted by $\Delta_1$ i.e.

$$\Delta_1=\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s+l,a\right)t^l$$

On using the definition (2.2.17) and changing the order of summation we have

$$\Delta_1=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}z^n}{(\sigma)_{\xi n}(a+n)^s\Gamma(\alpha n+\beta)n!}\left\{\sum_{l=0}^{\infty}\frac{(s)_l}{l!}\left(\frac{t}{a+n}\right)^l\right\}$$

Now on using the binomial expansion (2.2.19), we have

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n}}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)} \frac{1}{(a-t+n)^s}$$

Now on interpreting the resulting series in view of (2.2.17) we atonce arrive at the desired result in (2.5.1).

**Proof of (2.5.2)**

Let left hand side of (2.5.2) is denoted by $\Delta_2$ i.e.

$$\Delta_2 = \sum_{l=0}^{\infty} \binom{s+2l-1}{2l} \in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu} (z, s+2l, a) t^{2l}$$

On using the definition (2.2.17) and changing the order of summation we have

$$\Delta_2 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n} (a+n)^s \Gamma(\alpha+n\beta) n!} \left\{ \sum_{l=0}^{\infty} \frac{(s)_{2l}}{(2l)!} \left( \frac{t}{a+n} \right)^{2l} \right\}$$

Now in view of the following binomial expansion (2.2.20), it takes the following form

$$\Delta_2 = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)(a-t+n)^s} + \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)(a+t+n)^s} \right]$$

On interpreting the resulting series in view of definition (2.2.17), we atonce arrive at the desired result in (2.5.2).

**Proof of (2.5.3)**

Let left hand side of (2.5.3) is denoted by $\Delta_3$ i.e.

$$\Delta_3 = \sum_{l=0}^{\infty} \binom{s+2l}{2l+1} \in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu} (z, s+2l+1, a) t^{2l+1}$$

On using the definition (2.2.17) and changing the order of summation we have

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)(a+n)^s} \left\{ \sum_{l=0}^{\infty} \frac{(s)_{2l+1}}{(2l+1)!} \left( \frac{t}{a+n} \right)^{2l+1} \right\}$$

Now in view of the following binomial expansion (2.2.21), it takes the following form

$$\Delta_3 = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}\, z^n}{(\sigma)_{\xi n}\,\Gamma(\alpha n+\beta)(a-t+n)^s\, n!} - \sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}\, z^n}{(\sigma)_{\xi n}\,\Gamma(\alpha n+\beta)(a+t+n)^s\, n!}\right]$$

On interpreting the inner series in view of definition (2.2.17), we atonce arrive at the desired result in (2.5.3).

**Proof of (2.5.4)**

Let left hand side of (2.5.4) is denoted by $\Delta_4$ i.e.

$$\Delta_4 = \sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,v}\left(z,\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j+l,a\right)\frac{t^l}{l!}$$

On using the definition (2.2.17) and changing the order of summation we have

$$\Delta_4 = \sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}\, z^n}{(\sigma)_{\xi n}\, n!\,\Gamma(\alpha n+\beta)(a+n)^{\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j}}\left\{\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\left(\frac{t}{a+n}\right)^l\frac{1}{l!}\right\}$$

Now interpreting the inner series in generalized hypergeometric function ${}_pF_q$ in view of (2.2.22) we atonce arrive at the desired result in (2.5.4).

## 2.6 MILD EXTENSION AND ASSOCIATED GENERATING FUNCTIONS

A mild extension of the general function defined in (2.2.17) is also considered here, it is defined and represented in the following manner:

$$\in_{\alpha,\beta,(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a) = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(a+n)^s\,\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(2.6.1)$$

where $\forall(i=1,2,.......p; j=1,2,.......q)$, $p,q \in N_0$; $s,z,\mu_i,\alpha,\beta \in C$; $a,\sigma_j \in C/Z_0^-$; $\delta_i,\rho_j \in R^+$.

A Multi-parameter Mittag-Leffler function is also considered here it is defined as follows:

$$E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(2.6.2)$$

$\mu,\delta,\sigma,\rho,\alpha,\beta \in C$, $p,q \in N_0$ and $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$.

If we take $p=3$ *and* $q=2$; $\mu_1=\delta_1=1$ this reduce to the known Mittag-Leffler function studied by by Khan and Ahmed [47, p.2, eq. (1.9)].

The integral representation for the above function defined in (2.6.1) is given in the following theorem.

**Theorem-1**

For all $(i=1,2,.......p; j=1,2,.......q)$, $p,q \in N_0$; $s,z,\mu_i,\alpha,\beta \in C$;
$a,\sigma_j \in C/Z_0^-$; $\delta_i,\rho_j \in R^+$,
we have the following integral representation

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}e^{-at}t^{s-1}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(ze^{-t})dt \qquad \ldots(2.6.3)$$

where $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is the Mittag-Leffler function defined in (2.6.2):

The four generating functions involving general function defined in (2.6.1) are established here and given in the following theorems.

**Theorem-2**

For all $(i=1,2,.......p; j=1,2,.......q)$, $p,q \in N_0$; $s,z,\mu_i,\alpha,\beta \in C$;
$a,\sigma_j \in C/Z_0^-$; $\delta_i,\rho_j \in R^+$,
we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s}+\mathrm{l},a\right)t^{l} = \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s},a-\mathrm{t}\right) \qquad \ldots(2.6.4)$$

provided that both the side of (2.6.4) exists.

**Theorem-3**

For all $\left(i=1,2,.......\mathrm{p};\, j=1,2,.......\mathrm{q}\right)$, $p,q\in N_0$ ; $s,z,\mu_i,\alpha,\beta\in C$ ;

$a,\sigma_j\in C/Z_0^-$ ; $\delta_i,\rho_j\in R^+$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+2l-1}{2l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s}+2\mathrm{l},a\right)t^{2l}$$

$$=\frac{1}{2}\left[\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s},a+\mathrm{t}\right)+\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s},a-\mathrm{t}\right)\right] \qquad \ldots(2.6.5)$$

provided that both the side of (2.6.5) exists.

**Theorem-4**

For all $\left(i=1,2,.......\mathrm{p};\, j=1,2,.......\mathrm{q}\right)$, $p,q\in N_0$ ; $s,z,\mu_i,\alpha,\beta\in C$ ;

$a,\sigma_j\in C/Z_0^-$ ; $\delta_i,\rho_j\in R^+$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s}+2l+1,a\right)t^{2l+1}$$

$$=\frac{1}{2}\left[\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s},a-\mathrm{t}\right)-\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\mathrm{s},a+\mathrm{t}\right)\right] \qquad \ldots(2.6.6)$$

provided that both the side of (2.6.6) exists.

**Theorem-5**

For all $\left(i=1,2,.......\mathrm{p};\, j=1,2,.......\mathrm{q}\right)$, $p,q\in N_0$ ; $s,z,\mu_i,\alpha,\beta\in C$ ;

$a,\sigma_j\in C/Z_0^-$ ; $\delta_i,\rho_j\in R^+$, $\mathrm{Re}\left(\sum_{u=1}^{r}\theta_u\right)>\mathrm{Re}\left(\sum_{v=1}^{k}\lambda_v\right)>0$ ; $\left|\frac{t}{a}\right|<1$

we have the following generating relation

$$\sum_{l=0}^{\infty}\frac{\prod_{u=1}^{r}(\theta_u)_l}{\prod_{v=1}^{k}(\lambda_v)_l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\sum_{u=1}^{r}\theta_u-\sum_{v=1}^{k}\lambda_v+l,a\right)\frac{t^l}{l!}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}\left(\sigma_j\right)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\frac{1}{(a+n)^{\sum_{u=1}^{r}\theta_u-\sum_{v=1}^{k}\lambda_v}}\,{}_uF_v\left[\begin{matrix}(\theta_u);\dfrac{t}{a+n}\\(\lambda_v);a+n\end{matrix}\right] \qquad \ldots(2.6.7)$$

where ${}_pF_q$ is the generalized hypergeometric function defined in (2.2.22).

**OUTLINE OF PROOFS**

**Proof of (2.6.3)**

The function defined in (2.6.1) can be written straightly in the following form

$$\in_{\alpha,\beta,(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}\; z^n}{\prod_{j=1}^{q}\left(\sigma_j\right)_{\rho_j n}n!\Gamma(\alpha n+\beta)}\left\{\frac{\Gamma s}{(a+n)^s}\right\}$$

Now using the following elementary Gamma integral (2.2.18) and then changing the order of integration and summation, we have

$$\in_{\alpha,\beta,(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}e^{-at}t^{s-1}\left\{\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}\; z^n}{\prod_{j=1}^{q}\left(\sigma_j\right)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{\left(ze^{-t}\right)^n}{n!}\right\}dt$$

On interpreting the inner series in view of (2.6.2) we atonce arrive at the desired result (2.6.3).

**Proof of (2.6.4)**

Let left hand side of (2.6.4) is denoted by $\Delta_1$ i.e.

$$\Delta_1=\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s+l,a)t^l$$

On using the definition (2.6.1) and changing the order of summation we have

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} (a+n)^s \Gamma(\alpha n+\beta) n!} \left\{ \sum_{l=0}^{\infty} \frac{(s)_l}{l!} \left( \frac{t}{a+n} \right)^l \right\}$$

Now on using the binomial expansion (2.2.19), we have

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} \Gamma(\alpha n+\beta) n!} \frac{1}{(a-t+n)^s}$$

Now on interpreting the resulting series in view of (2.6.1) we atonce arrive at the desired result in (2.6.4).

**Proof of (2.6.5)**

Let left hand side of (2.6.5) is denoted by $\Delta_2$ i.e.

$$\Delta_2 = \sum_{l=0}^{\infty} \binom{s+2l-1}{2l} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)} (z, s+2l, a) t^{2l}$$

On using the definition (2.6.1) and changing the order of summation we have

$$\Delta_2 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} (a+n)^s \Gamma(\alpha n+\beta) n!} \left\{ \sum_{l=0}^{\infty} \frac{(s)_{2l}}{(2l)!} \left( \frac{t}{a+n} \right)^{2l} \right\}$$

Now in view of the following binomial expansion (2.2.20), it takes the following form

$$\Delta_2 = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} \; z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} n! \Gamma(\alpha n+\beta)(a-t+n)^s} + \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} \; z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} n! \Gamma(\alpha n+\beta)(a+t+n)^s} \right]$$

On interpreting the resulting series in view of definition (2.6.1), we at once arrive at the desired result in (2.6.5).

**Proof of (2.6.6)**

Let left hand side of (2.6.6) is denoted by $\Delta_3$ i.e.

$$\Delta_3 = \sum_{l=0}^{\infty} \binom{s+2l}{2l+1} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)} \left(z, s+2l+1, a\right) t^{2l+1}$$

On using the definition (2.6.1) and changing the order of summation we have

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} (a+n)^s \, \Gamma(\alpha n+\beta) n!} \left\{ \sum_{l=0}^{\infty} \frac{(s)_{2l+1}}{(2l+1)!} \left(\frac{t}{a+n}\right)^{2l+1} \right\}$$

Now in view of the following binomial expansion (2.2.21), it takes the following form

$$\Delta_3 = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} \; z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} n! \Gamma(\alpha n+\beta)(a-t+n)^s} - \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} \; z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} n! \Gamma(\alpha n+\beta)(a+t+n)^s} \right]$$

On interpreting the inner series in view of definition (2.6.1), we atonce arrive at the desired result in (2.6.6).

**Proof of (2.6.7)**

Let left hand side of (2.6.7) is denoted by $\Delta_4$ i.e.

$$\Delta_4 = \sum_{l=0}^{\infty} \frac{\prod_{u=1}^{r} (\theta_u)_l}{\prod_{v=1}^{k} (\lambda_v)_l} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)} \left( z, \sum_{u=1}^{r} \theta_u - \sum_{v=1}^{k} \lambda_v + l, a \right) \frac{t^l}{l!}$$

On using the definition (2.6.1) and changing the order of summation we have

$$\Delta_4 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\mu_i)_{\delta_i n} \; z^n}{\prod_{j=1}^{q} (\sigma_j)_{\rho_j n} n! \Gamma(\alpha n+\beta)(a+n)^{\sum_{u=1}^{r}\theta_u - \sum_{v=1}^{k}\lambda_v}} \left\{ \sum_{l=0}^{\infty} \frac{\prod_{u=1}^{r} (\theta_u)_l}{\prod_{v=1}^{k} (\lambda_v)_l} \left(\frac{t}{a+n}\right)^l \frac{1}{l!} \right\}$$

Now interpreting the inner series in generalized hypergeometric function ${}_pF_q$ in view of (2.2.22). we atonce arrive at the desired result in (2.6.7).

## 2.7 APPLICATIONS

The function studied in this paper are very general in nature and unify all the functions related to Hurwitz-Lerch Zeta function, therefore many known results can be obtained from our main results. As an application, some of these have been shown to derive as follows.

(i) If in the result (2.4.1), we apply $\mathrm{Lim}\,\alpha \to 0$ then it reduce to the known result due to Srivastava et. al [101, p. 494, eq. (2.4)].

(ii) If in result (2.5.1) to (2.5.4), we apply $\mathrm{Lim}\,\alpha \to 0$ then these results reduce to the known result due to Gupta [30, pp. 132-133, eqs. (2.9.1) to (2.9.4)] respectively.

(iii) If in result (2.6.3), we take $\mathrm{Lim}\,\alpha \to 0$ then it reduce to the integral representation for Hurwitz-Lerch Zeta function studied by Srivastava et. al [101, p.504, eq. (6.4)].

(iv) If we take $\mathrm{Lim}\,\alpha \to 0$ then result (2.6.4) to (2.6.7) reduce to known results for Hurwitz-Zeta function due to Gupta [30, pp. 134-135, eqs. (2.9.7) to (2.9.10)]

(v) If we take $\mu = \sigma, \delta = \xi, v = 1$ in result (2.5.1), it reduce to the known generating function due to Kumar ([56], p. 17, eq. (3.1) at $\delta = 1$).

(vi) For $p = 2, q = 1,\ \mu = \sigma,\ v = 1,\ \delta = \xi$ the result (2.5.4) provides the known result due to Kumar [56, p. 17, Theorem (3.2)].

## SECTION-B

**2.8** In direction to give more generalization of general function $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a\right)$ defined in (2.2.17), we consider the following multivariable analogue of this general function which is defined and represented in the following manner:

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}p_i n_i\right)^s\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left[\frac{\left(\mu_i\right)_{\delta_i n_i}z_i^{n_i}}{n_i!}\right]$$

…(2.8.1)

where $\forall\ i=1,2,...,r;\ \mu_i,\alpha_i,s,z\in C;\delta_i\in R^{+};\ \min\left\{\mathrm{Re}\left(a\right),\mathrm{Re}\left(s\right)\right\}>0;\ \max_{1\le i\le r}\left|z_i\right|<1,$

$\mathrm{Re}\left(\alpha_i\right)>0;\ \mathrm{Re}\left(p_i\right)>0.$

In this section, we established the integral representation of the general function defined in (2.8.1) and four generating relations involving this general function. As application of these results the integral representation and generating functions for simpler function related to Hurwitz-Lerch Zeta function are shown to be derived. The main results established in this section also provide the extensions of the results studied in Section-A of this chapter due to generalization of parameters and variables.

## 2.9 DEFINITIONS REQUIRED

The multivariable extension of Hurwitz-Lerch Zeta function (2.2.6) is defined and studied by Raina and Chhajed [80, p.98, eq.(5.1)]. we require this function at $b=0$ in the following manner:

$$\phi_{(\mu_i)}^{(p_i)}\left(x_1,...,x_n;\alpha,a\right)=\sum_{m_1,...,m_r=0}^{\infty}\left(a+\Omega\right)^{-\alpha}\prod_{i=1}^{n}\left[\frac{\left(\mu_i\right)_{m_i}x_i^{m_i}}{\left(m_i\right)}\right]\qquad\text{…(2.9.1)}$$

where $\mu\ge 1\left(i=1,2,...,n\right);\ \mathrm{Re}\left(\alpha\right)>0;\ \mathrm{Re}\left(p_i\right)>0;\ \max_{1\le i\le n}\left|x_i\right|<1;\quad \Omega=\sum_{i=1}^{n}p_i m_i$

A multivariable extension of Mittag-Leffler function defined in (2.2.10) is studied by Gautam [25] and Saxena et al. [88, p.536, eq. (1.14)], it is defined and represented as follows:

$$E_{(\rho_r),\lambda}^{(\gamma_r)}\left(z_1,...,z_r\right)=\sum_{k_1,...,k_r=0}^{\infty}\frac{\left(\gamma_1\right)_{k_1}...\left(\gamma_r\right)_{k_r}}{\Gamma\left(\lambda+k_1\rho_1+...+k_r\rho_r\right)}\frac{z_1^{k_1}...z_r^{k_r}}{k_1!,...,k_r!}\qquad\text{…(2.9.2)}$$

where $\lambda,\gamma_j,\rho_j \in C;\ \mathrm{Re}(\rho_j)>0;\ j=1,2,\ldots,r$

A mild generalization of multivariable analogue of Mittag-Leffler function (2.9.2) is also due to Saxena et.al [88, p.547, eq. (7.1)] (see also [30],[31]).

$$E_{(\rho_r),\lambda}^{(\gamma_r);(l_r)}(z_1,\ldots,z_r)=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1 l_1}\cdots(\gamma_r)_{k_r l_r}}{\Gamma(\lambda+k_1\rho_1+\ldots+k_r\rho_r)}\frac{z_1^{k_1}\ldots z_r^{k_r}}{k_1!\ldots k_r!} \qquad \ldots(2.9.3)$$

where $\lambda,\gamma_j,l_j,\rho_j \in C;\ ;\mathrm{Re}(\rho_j)>0;\ \mathrm{Re}(l_j)>0;\ \lambda\notin Z_0^-=\{0,-1,-2,\ldots\};\ j=1,2,\ldots,r$

The integral representation of the function (2.8.1) is obtained here.

## 2.10 INTEGRAL REPRESENTATION

For all $i=1\ldots r;\ \delta_i\in R^+,\ \mu_i,\alpha_i,s,z\in C;\ min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \mathrm{Re}(p_i)>0;$ $\mathrm{Re}(\alpha_i)>0;$ and $\max_{1\le i\le r}|z_i|<1$,

we have the following integral representation of the function defined in (2.8.1)

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}(z_1,\ldots,z_r;s,a)=\frac{1}{\Gamma s}\int_0^{\infty}t^{s-1}e^{-at}E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1e^{-p_1t},\ldots,z_re^{-p_rt}\right)dt \qquad \ldots(2.10.1)$$

where $E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}(z_1,\ldots,z_r)$ is the multivariable Mittag-Leffler function defined in (2.9.3).

**Proof of (2.10.1)**

We have the multivariable general function (2.8.1) in the following form,

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}(z_1,\ldots,z_r;s,a)=\frac{1}{\Gamma s}\sum_{n_1,\ldots,n_r=0}^{\infty}\left\{\frac{\Gamma s}{\left(a+\sum_{i=1}^{r}p_in_i\right)^s}\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\prod_{i=1}^{r}\left\{\frac{\left\{(\mu_i)_{\delta_in_i}\,z_i^{n_i}\right\}}{n_i!}\right\}$$

In view of (2.2.18), we have

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}(z_1,\ldots,z_r;s,a)=\frac{1}{\Gamma s}\sum_{n_1,\ldots,n_r=0}^{\infty}\left\{\int_0^{\infty}t^{s-1}e^{-\left(a+\sum_{i=1}^{r}p_in_i\right)t}dt\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\prod_{i=1}^{r}\left\{\frac{\left\{(\mu_i)_{\delta_in_i}\,z_i^{n_i}\right\}}{n_i!}\right\}$$

on interchanging the order of integration and summation it takes the following form

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left\{\frac{\left(\mu_i\right)_{\delta_i n_i}\left(z_i^{n_i}e^{-p_i t}\right)^{n_i}}{n_i\,!}\right\}$$

on interpreting the inner summation in view of (2.9.3), we atonce arrive at the required result given in (2.10.1).

## 2.11 GENERATING RELATIONS

In this section we establish four generating relations involving multivariable general function defined in (2.8.1).

**Theorem-1**

For all $i=1...\text{r}$; $\delta_i\in R^+$, $\mu_i,\alpha_i,\text{s},\text{z}\in C$; $min\left\{\text{Re}\left(a\right),\text{Re(s)}\right\}>0$; $\text{Re}\left(p_i\right)>0$;

$\text{Re}\left(\alpha_i\right)>0$; and $\max_{1\le i\le r}\left(|z_i|\right)<1$,

we have the following generating function

$$\sum_{i=0}^{\infty}\binom{s+l-1}{l}\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s+l,a\right)t^l=\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a-t\right)\qquad ...(2.11.1)$$

provided that both the side of (2.11.1) exists.

**Theorem-2**

For all $i=1...\text{r}$; $\delta_i\in R^+$, $\mu_i,\alpha_i,\text{s},\text{z}\in C$; $min\left\{\text{Re}\left(a\right),\text{Re(s)}\right\}>0$; $\text{Re}\left(p_i\right)>0$;

$\text{Re}\left(\alpha_i\right)>0$; and $\max_{1\le i\le r}\left(|z_i|\right)<1$,

we have the following generating function

$$\sum_{i=0}^{\infty}\binom{s+2l-1}{2l}\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s+2l,a\right)t^{2l}$$

$$=\frac{1}{2}\left[\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a+t\right)+\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a-t\right)\right]$$

...(2.11.2)

provided that both the side of (2.11.2) exists.

**Theorem-3**

For all $i = 1 \ldots \mathrm{r}$; $\delta_i \in R^+$ $\mu_i, \alpha_i, \mathrm{s}, \mathrm{z} \in C$; $min\{\mathrm{Re}(a), \mathrm{Re(s)}\} > 0$; $\mathrm{Re}(p_i) > 0$;

$\mathrm{Re}(\alpha_i) > 0$; and $\max\limits_{1 \le i \le r}(|z_i|) < 1$,

we have the following generating function

$$\sum_{i=0}^{\infty} \binom{s+2l}{2l+1} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} (z_1,\ldots,z_r; s+2l+1, a) t^{2l+1}$$

$$= \frac{1}{2}\left[\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} (z_1,\ldots,z_r; s, a-t) - \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} (z_1,\ldots,z_r; s, a+t)\right] \qquad \ldots(2.11.3)$$

provided that both the side of (2.11.3) exists.

**Theorem-4**

For all $\mathrm{Re}\left(\sum_{n=1}^{p} \theta_n\right) > \mathrm{Re}\left(\sum_{n=1}^{q} \gamma_n\right) > 0$ and $|t| < |a|$

we have the following generating relation

$$\sum_{l=0}^{\infty} \left( \frac{\prod_{n=1}^{p} (\theta_n)_l}{\prod_{n=1}^{q} (\gamma_n)_l} \right) \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left( z_1,\ldots,z_r; \sum_{n=1}^{p} \theta_n - \sum_{n=1}^{q} \gamma_n + l, a \right) \frac{t^l}{l!}$$

$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\Gamma\left(\beta + \sum_{i=1}^{r} \alpha_i n_i\right) \left(a + \sum_{i=1}^{r} p_i n_i\right)^{\sum_{n=1}^{p} \theta_n - \sum_{n=1}^{q} \gamma_n}} \prod_{i=1}^{r} \left[ \frac{(\mu_i)_{\delta_i n_i} \, z_i^{n_i}}{n_i!} \right] {}_pF_q \left[ \begin{matrix} (\theta_p); \\ (\gamma_q); \end{matrix} \frac{t}{\left(a + \sum_{i=1}^{r} p_i n_i\right)} \right]$$

$$\ldots(2.11.4)$$

where ${}_pF_q$ is the generalized hypergeometric function defined in (2.2.22).

## OUTLINE OF PROOFS

**Proof of (2.11.1)**

Let the left hand side of (2.11.1) denoted by $\Delta_1$ i.e.

$$\Delta_1 = \sum_{l=0}^{\infty} \binom{s+l-1}{l} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} (z_1,\ldots,z_r; s+l, a) t^l$$

On using the definition (2.8.1), we have

$$\Delta_1 = \sum_{l=0}^{\infty} \frac{(s)_l}{l!} \left[ \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^{s+l} \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{\delta_i n_i} z_i^{n_i}}{n_i!} \right\} \right] t^l$$

on interchanging the order of summation, we have

$$\Delta_1 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^{s} \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{\delta_i n_i} z_i^{n_i}}{n_i!} \right\} \left\{ \sum_{l=0}^{\infty} \frac{(s)_l}{l!} \left( \frac{t}{a+\sum_{i=1}^{r} p_i n_i} \right)^l \right\}$$

Now in view of (2.2.19) and then interpreting the multiple series with the help of (2.8.1) we atonce arrive at the required result in (2.11.1).

**Proof of (2.11.2)**

Let the left hand side of (2.11.2) and denoted by $\Delta_2$ i.e.

$$\Delta_2 = \sum_{l=0}^{\infty} \binom{s+2l-1}{2l} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(z_1,\ldots,z_r; s+2l, a\right) t^{2l}$$

On using the definition (2.8.1), we have

$$\Delta_2 = \sum_{l=0}^{\infty} \frac{(s)_{2l}}{(2l)!} \left[ \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^{s+2l} \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{\delta_i n_i} z_i^{n_i}}{n_i!} \right\} \right] t^{2l}$$

on interchanging the order of summation, we have

$$\Delta_2 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^{s} \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{\delta_i n_i} z_i^{n_i}}{n_i!} \right\} \left\{ \sum_{l=0}^{\infty} \frac{(s)_{2l}}{(2l)!} \left( \frac{t}{a+\sum_{i=1}^{r} p_i n_i} \right)^{2l} \right\}$$

On using the relation (2.2.20), it takes the following form

$$\Delta_2 = \sum_{n_1,..,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{(\mu_i)_{\delta_i n_i}\, z_i^{n_i}}{n_i\,!}\right\}$$

$$\left[\frac{1}{2}\left\{\left(1-\frac{t}{a+\sum_{i=1}^{r} p_i n_i}\right)^{-s} + \left(1+\frac{t}{a+\sum_{i=1}^{r} p_i n_i}\right)^{-s}\right\}\right]$$

on simplifying and then interpreting the multiple series with the help of (2.8.1), we atonce arrive at the desired result in (2.11.2).

**Proof of (2.11.3)**

Let the left hand side of (2.11.3) denoted by $\Delta_3$ i.e.

$$\Delta_3 = \sum_{l=0}^{\infty}\binom{s+2l}{2l+1} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(z_1,...,z_r; s+2l+1, a\right) t^{2l+1}$$

On using the definition (2.8.1) and then changing the order of summation, we have

$$\Delta_3 = \sum_{n_1,...,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{(\mu_i)_{\delta_i n_i}\, z_i^{n_i}}{n_i\,!}\right\}\left(\sum_{l=0}^{\infty}\frac{(s)_{2l+1}}{(2l+1)!}\left(\frac{t}{a+\sum_{i=1}^{r} p_i n_i}\right)^{2l+1}\right)$$

in view of the (2.2.21), it takes the following form

$$\Delta_3 = \sum_{n_1,...,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{(\mu_i)_{\delta_i n_i}\, z_i^{n_i}}{n_i\,!}\right\}$$

$$\left[\frac{1}{2}\left\{\left(1-\frac{t}{a+\sum_{i=1}^{r} p_i n_i}\right)^{-s} - \left(1+\frac{t}{a+\sum_{i=1}^{r} p_i n_i}\right)^{-s}\right\}\right]$$

on simplifying and then interpreting the multiple series in view of definition (2.8.1), we atonce arrive at the desired result in (2.11.3).

**Proof of (2.11.4)**

Let the left hand side of (2.11.4) is denoted by $\Delta_4$ i.e.

$$\Delta_4 = \sum_{l=0}^{\infty} \left( \frac{\prod_{n=1}^{p} (\theta_n)_l}{\prod_{n=1}^{q} (\gamma_n)_l} \right) \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left( z_1,\ldots,z_r; \sum_{n=1}^{p}\theta_n - \sum_{n=1}^{q}\gamma_n + l, a \right) \frac{t^l}{l!}$$

On using the definition (2.8.1) and changing the order of summation, the above equation takes the following form

$$\Delta_4 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^{\sum_{n=1}^{p}\theta_n - \sum_{n=1}^{q}\gamma_n} \Gamma\left(\beta + \sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r} \left\{ (\mu_i)_{\delta_i n_i} \frac{z_i^{n_i}}{n_i!} \right\}$$

$$\left[ \sum_{l=0}^{\infty} \frac{\prod_{n=1}^{p} (\theta_n)_l}{\prod_{n=1}^{q} (\gamma_n)_l} \left( \frac{t}{a+\sum_{i=1}^{r} p_i n_i} \right)^l \frac{1}{l!} \right]$$

Now interpreting the inner series in view of the definition of generalized hypergeometric function (2.2.22), we atonce arrive at the desired result in (2.11.4).

## 2.12 MILD EXTENSION AND RELATED GENERATING RELATIONS

Now we consider a mild extension of general function defined in (2.8.1) in view of the parameters. It is defined and represented in the following manner: $\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)} \left( z_1,\ldots,z_r; s, a \right)$

$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^{s} \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r} \left[ \frac{\prod_{j=1}^{p_i} \left(\mu_j^{(i)}\right)_{\delta_j^{(i)} n_i}}{\prod_{k=1}^{q_i} \left(\sigma_k^{(i)}\right)_{\rho_k^{(i)} n_i}} \frac{z_i^{n_i}}{n_i!} \right] \qquad \ldots(2.12.1)$$

where $\forall\ i=1\ldots r;\ p_i, q_i \in N_0;\ \delta_j^{(i)}, \rho_k^{(i)} \in R^+;\ a, \sigma_k^{(i)} \in C/Z_0^-;\quad \gamma_i, s, z, \mu_j^{(i)}, \alpha_i, \beta \in C$ $\left(j=1\ldots p_i; k=1\ldots q_i\right)$, and for all $i=1\ldots r$, the notation $(\mu_j^{(i)}) \equiv \mu_1^{(i)}, \ldots\ldots, \mu_{p_i}^{(i)}$ and $(\delta_j^{(i)})$ abbreviate the same array of parameters as in $(\mu_j^{(i)})$. $(\sigma_k^{(i)}) \equiv \sigma_1^{(i)}, \ldots, \sigma_{q_i}^{(i)}$ and $(\rho_k^{(i)})$ abbreviate the same array of parameters as in $(\sigma_k^{(i)})$ and $(\alpha_i) \equiv \alpha_1, \ldots, \alpha_r; (\gamma_i) \equiv \gamma_1, \ldots, \gamma_r$.

We consider here the following extension of multivariable Mittag-Leffler function (2.9.3) this extended multivariable function is represented and defined in the following manner:

$$E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1, \ldots, z_r\right) = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\Gamma\left(\beta + \sum_{i=1}^{r} \alpha_i n_i\right)} \prod_{i=1}^{r} \left[ \frac{\prod_{j=1}^{p_i} \left(\mu_j^{(i)}\right)_{\delta_j^{(i)} n_i}}{\prod_{k=1}^{q_i} \left(\sigma_k^{(i)}\right)_{\rho_k^{(i)} n_i}} \frac{z_i^{n_i}}{n_i!} \right] \qquad \ldots(2.12.2)$$

$\forall (i=1,\ldots,r;\ j=1,\ldots,p_i; k=1,\ldots,q)$, $\mu_j^{(i)}, \delta_j^{(i)}, \sigma_k^{(i)}, \rho_k^{(i)}, \alpha_i, \beta \in C,\ \ p_i, q_i \in N_0$ and

$\min\left\{\mathrm{Re}(\mu_j^{(i)}), \mathrm{Re}(\alpha_i), \mathrm{Re}(\beta), \mathrm{Re}(\mu_j^{(i)}), \mathrm{Re}(\delta_j^{(i)}), \mathrm{Re}(\sigma_k^{(i)}), \mathrm{Re}(\rho_k^{(i)})\right\} > 0$.

the notation $(\mu_j^{(i)}) \equiv \mu_1^{(i)}, \ldots, \mu_{p_i}^{(i)}$ and $(\delta_j^{(i)})$ abbreviate the same array of parameters as in $(\mu_j^{(i)})$. $(\sigma_k^{(i)}) \equiv \sigma_1^{(i)}, \ldots, \sigma_{q_i}^{(i)}$ and $(\rho_k^{(i)})$ abbreviate the same array of parameters as in $(\sigma_k^{(i)})$ and $(\alpha_i) \equiv \alpha_1, \ldots, \alpha_r$.

**Remark:** At $p_i = q_i = 1\ (\forall i = 1, \ldots.. r)$ the extended multivariable function (2.12.2) reduce to the multivariable Mittag-Leffler function due to Gurjar [33, p.7, eq. (1.8)].

The integral representation for the above general function defined in (2.12.1) is given in the following theorem.

**Theorem-1**

For all $i=1\ldots r;\quad j=1\ldots p_i;\ k=1,..,q_i;\ p_i, q_i \in N_0;\quad \delta_j^{(i)}, \rho_k^{(i)} \in R^+;\quad a, \sigma_k^{(i)} \in C/Z_0^-$

$\gamma_i, s, z, \mu_j^{(i)}, \alpha_i, \beta \in C$,

we have the following integral representation of the function defined in (2.12.1)

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_re^{-\gamma_r t}\right)dt$$

...(2.12.3)

where $E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r\right)$ is the extended of multivariable Mittag-Leffler function defined in (2.12.2).

The four generating relations involving multivariable general function defined in (2.12.1) are also established here and given as the following theorems:

**Theorem-2**

For all $\left(i=1...\text{r};\ j=1...p_i;\ \text{k}=1...\text{q}_i\right),\ p_i,\text{q}_i\in N_0;\ \delta_j^{(i)},\rho_k^{(i)}\in R^+;\ a,\sigma_k^{(i)}\in C/Z_0^-$

$\gamma_i, s, \text{z}, \mu_j^{(i)}, \alpha_i, \beta\in C$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s+l,a\right)t^l=\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a-t\right)$$

…(2.12.4)

provided that both the side of (2.12.4) exists.

**Theorem-3**

For all $\left(i=1...\text{r};\ j=1...p_i;\ \text{k}=1...\text{q}_i\right),\ p_i,\text{q}_i\in N_0;\ \delta_j^{(i)},\rho_k^{(i)}\in R^+;\ a,\sigma_k^{(i)}\in C/Z_0^-$

$\gamma_i, s, \text{z}, \mu_j^{(i)}, \alpha_i, \beta\in C$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+2l-1}{2l}\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s+2l,a\right)t^{2l}$$

$$=\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a+t\right)+\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a-t\right)\qquad ..(2.12.5)$$

provided that both the side of (2.12.5) exists.

**Theorem-4**

For all $\left(i=1...\mathrm{r};\ j=1...p_i;\ \mathrm{k}=1...\mathrm{q}_i\right),\ p_i,\mathrm{q}_i\in N_0;\ \delta_j^{(i)},\rho_k^{(i)}\in R^+;\ a,\sigma_k^{(i)}\in C/Z_0^-$

$\gamma_i, s, \mathrm{z}, \mu_j^{(i)}, \alpha_i, \beta \in C$,

we have the following generating function

$$\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{(\alpha_i),\beta;\gamma_i,\left(\sigma_k^{(i)}\right)\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right),\left(\delta_j^{(i)}\right)}\left(z_1,...,\mathrm{z}_r;s+2l+1,a\right)t^{2l+1}$$

$$=\left[\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,\mathrm{z}_r;s,a-t\right)-\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,\mathrm{z}_r;s,a+t\right)\right] \quad ...(2.12.6)$$

provided that both the side of (2.12.6) exists.

**Theorem-5**

For $\mathrm{Re}\left(\sum_{n=1}^{p}\theta_n\right)>\mathrm{Re}\left(\sum_{n=1}^{q}v_n\right)>0$ and $|t|<|a|$

we have the following relation

$$\sum_{l=0}^{\infty}\frac{\prod_{n=1}^{p}(\theta_n)_l}{\prod_{n=1}^{q}(v_n)_l}\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,\mathrm{z}_r;\sum_{n=1}^{p}\theta_n-\sum_{n=1}^{q}v_n+l,a\right)\frac{t^l}{l!}$$

$$=\sum_{n_1,...,\mathrm{n}_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^{\sum_{n=1}^{p}\theta_n-\sum_{n=1}^{q}v_n}\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i!}\right]{}_pF_q\left[\begin{matrix}(\theta_p); \\ (v_q);\end{matrix}\ \frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right]$$

…(2.12.7)

where ${}_pF_q$ is the generalized hypergeometric function defined in (2.2.22).

**OUTLINE OF PROOFS**

**Proof of (2.12.3)**

We have the multivariable general function (2.12.1) in the following form,

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,\mathrm{z}_r;s,a\right)=\frac{1}{\Gamma s}\sum_{n_1,...,\mathrm{n}_r=0}^{\infty}\left\{\frac{\Gamma s}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s}\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}$$

$$\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i\,!}\right]$$

In view of (2.2.18), we have

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{n_1,...,n_r=0}^{\infty}\left\{\int_0^\infty t^{s-1}e^{-\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)t}dt\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}$$

$$\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i\,!}\right]$$

on interchanging the order of integration and summation it takes the following form

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}\left(z_i^{n_i}e^{-\gamma_i t}\right)^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}\; n_i\,!}\right\}$$

on interpreting the inner summation in view of (2.12.2), we atonce arrive at the required result given in (2.12.3).

**Proof of (2.12.4)**

Let the left hand side of (2.12.4) denoted by $\Delta_1$ i.e.

$$\Delta_1=\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s+l,a\right)t^l$$

On using the definition (2.12.1), we have

$$\Delta_1=\sum_{l=0}^{\infty}\frac{(s)_l}{l!}\left[\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^{s+l}\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i\,!}\right\}\right]t^l$$

on interchanging the order of summation, we have

$$\Delta_1 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i\,!}\right\}\left\{\sum_{l=0}^{\infty}\frac{(s)_l}{l!}\left(\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^l\right\}$$

Now in view of (2.2.19) and then interpreting the multiple series with the help of (2.12.1) we atonce arrive at the required result in (2.12.4).

**Proof of (2.12.5)**

Let the left hand side of (2.12.5) and denoted by $\Delta_2$ i.e.

$$\Delta_2 = \sum_{l=0}^{\infty}\binom{s+2l-1}{2l} \in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,\ldots,z_r;s+2l,a\right)t^{2l}$$

On using the definition (2.12.1), we have

$$\Delta_2 = \sum_{l=0}^{\infty}\frac{(s)_{2l}}{(2l)!}\left[\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^{s+2l} \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i} z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i} n_i\,!}\right\}\right]t^{2l}$$

on interchanging the order of summation, we have

$$\Delta_2 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i} z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i} n_i\,!}\right\}\left\{\sum_{l=0}^{\infty}\frac{(s)_{2l}}{(2l)!}\left(\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^{2l}\right\}$$

On using the relation (2.2.20), it takes the following form

$$\Delta_2 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i} z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i} n_i!}\right\}$$

$$\times \left[\frac{1}{2}\left\{\left(1-\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^{-s}+\left(1+\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^{-s}\right\}\right]$$

on simplifying and then interpreting the multiple series with the help of (2.12.1), we atonce arrive at the desired result in (2.12.5).

**Proof of (2.12.6)**

Let the left hand side of (2.12.6) denoted by $\Delta_3$ i.e.

$$\Delta_3 = \sum_{l=0}^{\infty}\binom{s+2l}{2l+1} \in_{(\alpha_i),\beta;\gamma_i,\left(\sigma_k^{(i)}\right)\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right),\left(\delta_j^{(i)}\right)} \left(z_1,\ldots,z_r;s+2l+1,a\right)t^{2l+1}$$

On using the definition (2.12.1) and then changing the order of summation, we have

$$\Delta_3 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i} z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i} n_i!}\right\}$$

$$\left(\sum_{l=0}^{\infty}\frac{(s)_{2l+1}}{(2l+1)!}\left(\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^{2l+1}\right)$$

in view of the (2.2.21), it takes the following form

$$\Delta_3 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i} z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i} n_i!}\right\}$$

$$\times \left[\frac{1}{2}\left\{\left(1-\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^{-s}-\left(1+\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^{-s}\right\}\right]$$

on simplifying and then interpreting the multiple series in view of definition (2.12.1), we at once arrive at the desired result in (2.12.6).

**Proof of (2.12.7)**

Let the left hand side of (2.12.7) is denoted by $\Delta_4$ i.e.

$$\Delta_4 = \sum_{l=0}^{\infty} \frac{\prod_{n=1}^{p}(\theta_n)_l}{\prod_{n=1}^{q}(v_n)_l} \in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)} \left(z_1,\ldots,z_r; \sum_{n=1}^{p}\theta_n - \sum_{n=1}^{q} v_n + l, a\right)\frac{t^l}{l!}$$

On using the definition (2.12.1) and changing the order of summation, the above equation takes the following form

$$\Delta_4 = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^{\sum_{n=1}^{p}\theta_n - \sum_{n=1}^{q}v_n} \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)} n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)} n_i}} \frac{z_i^{n_i}}{n_i!}\right]$$

$$\times \left[\sum_{l=0}^{\infty} \frac{\prod_{n=1}^{p}(\theta_n)_l}{\prod_{n=1}^{q}(v_n)_l}\left(\frac{t}{a+\sum_{i=1}^{r}\gamma_i n_i}\right)^l \frac{1}{l!}\right]$$

Now interpreting the inner series in view of the definition of generalized hypergeometric function (2.2.22), we at once arrive at the desired result in (2.12.7).

## 2.13 APPLICATIONS

The multivariable general function defined in (2.8.1) and its mild extension defined in (2.12.1) unify all the functions associated to Hurwitz-Lerch Zeta function of one and several variables. Therefore, as the applications, many new and known results can be obtained from our main results. Some of these have been shown to derive as follows.

(i) If in result (2.12.3), we apply $\text{Lim}\,\alpha_i \to 0$ and take $\delta_i = 1$ then it reduce to the known result due to Raina and Chhajed ([80],p.99,eq.(5.2) at $b=0$) which in turn at $r = p_i = 1$ provides the known result due to Goyal and Laddha [28, p.100, eq.(1.6)].

(ii) If in results (2.11.1), (2.11.2), (2.11.3) and (2.11.4), we apply $\mathrm{Lim}\,\alpha_i \to 0$ and take $\delta_i = 1$ then these results provide the following corresponding new generating relations for multivariable Hurwitz-Lerch Zeta function $\phi_{(\mu_1)}^{(p_i)}\left(z_1,...,z_r;s,a\right)$ defined in (2.9.1) due to Raina and Chhajed ([80],p.98, eq.(5.1) at $b=0$).

**Corollary-1**

For all $i = 1...r$; $\mu_i, s, z \in C$; $min\left\{\mathrm{Re}(a), \mathrm{Re}(s)\right\} > 0$; $\mathrm{Re}(p_i) > 0$ and

$\max\limits_{1\le i\le r}\left(|z_i|\right) < 1$

we have the following generating relation,

$$\sum_{i=0}^{\infty}\binom{s+l-1}{l}\phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s+l,a\right)t^l = \phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s,a-t\right) \qquad \ldots(2.13.1)$$

provided that both the side of (2.13.1) exists.

**Corollary-2**

For all $i = 1...r$; $\mu_i, s, z \in C$; $min\left\{\mathrm{Re}(a), \mathrm{Re}(s)\right\} > 0$; $\mathrm{Re}(p_i) > 0$; and

$\max\limits_{1\le i\le r}\left(|z_i|\right) < 1$

we have the following generating relation,

$$\sum_{i=0}^{\infty}\binom{s+2l-1}{2l}\phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s+2l,a\right)t^{2l}$$

$$= \frac{1}{2}\left[\phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s,a+t\right) + \phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s,a-t\right)\right] \qquad \ldots(2.13.2)$$

provided that both the side of (2.13.2) exists.

**Corollary-3**

For all $i = 1...r$; $\mu_i, s, z \in C$; $min\left\{\mathrm{Re}(a), \mathrm{Re}(s)\right\} > 0$; $\mathrm{Re}(p_i) > 0$; and

$\max\limits_{1\le i\le r}\left(|z_i|\right) < 1$

we have the following generating relation,

$$\sum_{i=0}^{\infty}\binom{s+2l}{2l}\phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s+2l+1,a\right)t^{2l+1}$$

$$= \frac{1}{2}\left[\phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s,a-t\right) - \phi_{(\mu_r)}^{(p_r)}\left(z_1,...,z_r;s,a+t\right)\right] \qquad \ldots(2.13.3)$$

provided that both the side of (2.13.3) exists.

**Corollary-4**

For all $\operatorname{Re}\left(\sum_{n=1}^{p}\theta_n\right) > \operatorname{Re}\left(\sum_{n=1}^{q}\gamma_n\right) > 0$ and $|t| < |a|$

The following generating relation holds good,

$$\sum_{l=0}^{\infty}\left(\frac{\prod_{n=1}^{p}(\theta_n)_l}{\prod_{n=1}^{q}(\gamma_n)_l}\right)\phi_{(\mu_r)}^{(p_r)}\left(z_1,\dots,z_r;\sum_{n=1}^{p}\theta_n-\sum_{n=1}^{q}\gamma_n+l,a\right)\frac{t^l}{l!}$$

$$=\sum_{n_1,\dots,n_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}p_in_i\right)^{\sum_{n=1}^{p}\theta_n-\sum_{n=1}^{q}\gamma_n}}\prod_{i=1}^{r}\left[\frac{(\mu_i)_{n_i}\,z_i^{n_i}}{n_i!}\right]{}_pF_q\left[\begin{matrix}(\theta_p); \\ (\gamma_q);\end{matrix}\frac{t}{\left(a+\sum_{i=1}^{r}p_in_i\right)}\right]$$

…(2.13.4)

where ${}_pF_q$ is the generalized hypergeometric function defined in (2.2.22).

(iii) For $r=1$, $p=2$, $q=1$, $\operatorname{Lim}\alpha_1\to 0$, $\delta_1=1$ and $p_1=1$, the result (2.11.4) reduce to the known result due to Raina and Chhajed ([80],p.95,eq.(3.9) at $b=0$).

(iv) For $\operatorname{Lim}\alpha_1\to 0$,, results (2.12.3), (2.12.4), (2.12.5), (2.12.6), and (2.12.7) reduce to the known results due to Gupta ([30],p.110, eq.(2.6.2); pp.120-121, eqs.(2.8.7),(2.8.8),(2.8.9),(2.8.10) at $b=0$) respectively.

(v) For $r=1$, $p_1=2$, $q_1=1$, $\gamma_1=1$, results (2.12.3), (2.12.4), (2.12.5), (2.12.6) and (2.12.7) reduce to the known results due to Jaimini and Tatwal [43, pp.954-956, Theorem 1.1, Theorem 2.1 to 2.4] respectively.

(vi) For $r=1$ and $\gamma_1=1$, results (2.12.3), (2.12.4), (2.12.5), (2.12.6) and (2.12.7) reduce to the known results due to Jaimini and Tatwal [43 , pp.959-960, Theorem 3.1 to 3.5] respectively.

## 2.14 SPECIAL REMARK

The results obtained in this chapter are related with the Hurwitz Lerch Zeta functions of single and multivariable that will be useful to solve a variety of problems of fractional kinetic equations, integral transforms, integral representations and generating relations. The present study is worthwhile as the properties of Mittag-Leffler function and their generalizations are being study [86,93,97]. The Mittag-Leffler function is widely used to solve differential equation of fractional order [42,44].

Our proposed study will be very useful and provide the new direction of research for young researcher.

# CHAPTER-3

# INTEGRAL REPRESENTATIONS AND INTEGRAL EXPRESSIONS ASSOCIATED TO THE GENERAL FUNCTIONS OF ONE AND SEVERAL VARIABLES IN TERMS OF MITTAG-LEFFLER FUNCTIONS

The main results of this chapter have been published / accepted / communicated for publication as per details below:

1 Certain integral representation and fractional derivatives associated to a general function related to Hurwitz-Lerch Zeta function.
Panamerican Mathematical Journal Vol. 35, No. 4s (2025), 1240-1250

2 Multivariable analogue of a function associated with the Hurwitz-Lerch Zeta function and its integral expressions in term of Mittag- Leffler functions of several variables.
(Communicated for publication)

## 3.1 INTRODUCTION

In this chapter, we have explored certain integral representation and fractional derivatives and certain integral expressions of the general class of functions of one and several variables. This chapter is structured into two parts: Section-A and Section-B. In Section-A the integral representations and fractional derivative for univariate general function are established, while in Section-B, we consider the multivariable extension of this general function and certain integral expressions are obtained for the multivariable analogue of the general function studied in Section-A.

Further mild extensions for these general functions of single and multivariable are also considered in this chapter. The integral representations and fractional derivative for this extended function of single variable are studied in Section-A and the certain integral expressions for the extended function of multivariable are investigated in Section-B.

In Section-A, we have evaluated four integral representations and one fractional derivative for the general function of single variable and these results are further studied for its mild extension. As applications of integral representations and fractional derivative established in Section-A, the integral representations and fractional derivatives for simpler function related to the Hurwitz-Lerch Zeta function are shown to be derived.

In Section-B, we studied a multivariable analogue of general function and its mild extension in view of parameters. we have obtained integral representation and three series integral expressions for this multivariable general function and for its mild extension in terms of Mittag-Leffler function of several variable. Some known and new results are shown to be obtained for the Hurwitz-Lerch Zeta function as application of our main results established here.

## 3.2 DEFINITIONS AND PRILIMINARIES

A general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in the following form [19, p.27, eq. (1.11.1)]:

$$\phi(z,\mathrm{s},a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \qquad \ldots(3.2.1)$$

$a\in C/Z_0^-, \mathrm{s}\in C$ when $|z|<1$ and $\mathrm{Re}(s)>1$ when $|z|=1$)

The Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in (3.2.1) have a special case, as the Riemann Zeta function $\zeta(s)$ and Lerch Zeta function $\zeta(s,a)$ defined by following manner [19, Chapter-1] [98, Chapter-2]

$$\zeta(s)=\sum_{n=0}^{\infty}\frac{1}{(n+1)^s}=\phi(1,s,1) \qquad (\mathrm{Re}(s)>1) \qquad \ldots(3.2.2)$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\phi(1,s,a) \qquad (\mathrm{Re}(s)>1; a\in C/Z_0^-) \qquad \ldots(3.2.3)$$

An extension of the Hurwitz-Lerch Zeta function was investigated and studied by Lin and Srivastava in following manner [58, p.727, eq. (8)]

$$\phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s} \qquad \ldots(3.2.4)$$

$(\mu\in C; a,\nu\in C/Z_0^-, \rho,\sigma\in R^+; \rho<\sigma$ when $s,z\in C; \rho=\sigma$ and $s\in C$ when $|z|<\delta=\rho^{-\rho}\sigma^{\sigma}; \rho=\sigma$ and $\mathrm{Re}(s-\mu-\nu)>1$ when $|z|=\delta)$

where $(\lambda)_n$ denotes the general pochhammer symbol which is defined by

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma\gamma} \qquad \ldots(3.2.5)$$

$$(\gamma)_0=1, (\gamma)_n=\gamma(\gamma+1).....(\gamma+n-1)$$

where $n=1,2,....$

We can find from (3.2.4) that

$$\phi_{(\mu,1)}^{(1,1)}(z,s,a)=\phi_{\mu}^{*}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(a+n)^s} \qquad \ldots(3.2.6)$$

$(\mu\in C; a\in C/Z_0^-, s\in C$ when $|z|<1$ and $\mathrm{Re}(s-\mu)>1$ when $|z|=1)$

Where $\phi_{\mu}^{*}(z,s,a)$ is the Hurwitz-Lerch Zeta function considered by Goyal and Laddha [28, p.100, eq. (1.5)].

A generalization of Hurwitz-Lerch Zeta function (3.2.1) and (3.2.6) was defined and investigated by Garg et. al [23, p.313, eq. (1.7)]:

$$\phi_{\lambda,\mu;\nu}\left(z,s,a\right)=\sum_{n=0}^{\infty}\frac{\left(\lambda\right)_n\left(\mu\right)_n}{\left(\nu\right)_n n!}\frac{z^n}{\left(n+a\right)^s} \qquad \ldots(3.2.7)$$

$(\lambda,\mu\in C;\nu,a\in C/Z_0^-;s\in C$ when $|z|<1$ and $\mathrm{Re}\left(s+\nu-\lambda-\mu\right)>1$ when $|z|=1)$

A further generalization and investigation of a new family of Hurwitz-Lerch Zeta function defined by Srivastava and Pogany [101, p.491, eq. (1.20)] in the following form:

$$\phi_{\lambda,\mu;\nu}^{\rho,\sigma,\mathrm{k}}\left(z,s,a\right)=\sum_{n=0}^{\infty}\frac{\left(\lambda\right)_{\rho n}\left(\mu\right)_{\sigma n}}{\left(\nu\right)_{kn} n!}\frac{z^n}{\left(a+n\right)^s} \qquad \ldots(3.2.8)$$

$(\lambda,\mu\in C;\ \nu,a\in C/Z_0^-;\ \rho,\sigma,\mathrm{k}\in\mathrm{R}^+;\ k-\rho-\sigma>-1$ when $s,z\in C;k-\rho-\sigma=-1$

and $s\in C$ when $|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}\mathrm{k}^k;\ k-\rho-\sigma=-1$ and $\mathrm{Re}\left(s+\nu-\lambda-\mu\right)>1$ when $|z|=\delta^*)$

The well known Mittag-Leffler function $E_\alpha\left(z\right)$ is defined by following form [66] (See also [67]):

$$E_\alpha\left(z\right)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\alpha n+1\right)},\ \left(\mathrm{z}\in C,\mathrm{Re}\left(\alpha\right)>0\right) \qquad \ldots(3.2.9)$$

A generalization of Mittag-Leffler function is introduced by Prabhakar [78] as follows:

$$E_{\alpha,\beta}^{\gamma}\left(z\right)=\sum_{n=0}^{\infty}\frac{\left(\gamma\right)_n z^n}{\Gamma\left(\alpha n+\beta\right)n!} \qquad \ldots(3.2.10)$$

where $\alpha,\beta,\gamma\in C;\mathrm{Re}\left(\alpha\right)>0;\mathrm{Re}\left(\beta\right)>0;\mathrm{z}\in C$

This function further generalized by Srivastava and Tomovski [97] as following form (see also [86],[93]):

$$E_{\alpha,\beta}^{\gamma,k}\left(z\right)=\sum_{n=0}^{\infty}\frac{\left(\gamma\right)_{kn}}{\Gamma\left(\alpha n+\beta\right)}\frac{z^n}{n!} \qquad \ldots(3.2.11)$$

where $\mathrm{z},\beta,\gamma,\mathrm{k}\in C;\mathrm{Re}\left(\alpha\right)>Max\left\{0,\mathrm{Re}\left(k\right)-1\right\};\mathrm{Re}\left(k\right)>0$

In view of extension of parameters, Mittag-Leffler function was also studied by Khan et. al. [47, p.2, eq. (1.9)]:

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\eta,\mathrm{q}}\left(z\right)=\sum_{n=0}^{\infty}\frac{\left(\mu\right)_{\rho n}\left(\eta\right)_{qn}}{\left(\nu\right)_{\sigma n}\left(\delta\right)_{pn}}\frac{z^n}{\Gamma\left(\alpha n+\beta\right)} \qquad \ldots(3.2.12)$$

where $\alpha,\beta,\eta,\delta,\mu,\nu,\rho,\sigma\in C;\ p,q>0$, $\mathrm{q}\le \mathrm{Re}(\alpha)+p$ and

$\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\delta),\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$

The function (3.2.12) at $\delta=p=1$ takes the following form

$$E_{\alpha,\beta,\nu,\sigma}^{\mu,\rho,\eta,\mathrm{q}}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}\,z^n}{(\nu)_{\sigma n}\,n!\ \Gamma(\alpha n+\beta)} \qquad \ldots(3.2.13)$$

where $\alpha,\beta,\eta,\nu,\rho,\sigma\in C;$ and $\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$

In 1940, Wright Introduced and studied the following Taylor-Maclaurin Series [114, p.424]:

$$\in_{\alpha,\beta}(\phi;z)=\sum_{n=0}^{\infty}\frac{\phi(n)\,z^n}{\Gamma(\alpha n+\beta)} \qquad \ldots(3.2.14)$$

$$\left(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0\right)$$

where $\phi(n)$ is a function satisfying suitable conditions.

Motivated by the work of Wright [114], E.W. Barnes [4] considered the asymptotic expansion of function in the class which is defined as:

$$E_{\alpha,\beta}^{(k)}(\mathrm{s},z)=\sum_{n=0}^{\infty}\frac{z^n}{(n+k)^s\,\Gamma(\alpha n+\beta)} \qquad \ldots(3.2.15)$$

$$\left(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0\right)$$

for suitable restricted parameters $k$ and $s$.

Recently for suitably restricted sequence $\{\phi(n)\}_{n=0}^{\infty}$, Srivastava [94] introduced a class of function (See also [105],[110]):

$$\in_{\alpha,\beta}(\phi;z,s,a)=\sum_{n=0}^{\infty}\frac{\phi(n)\,z^n}{(a+n)^s\,\Gamma(\alpha n+\beta)} \qquad \ldots(3.2.16)$$

$$\left(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0\right)$$

Further for $\phi(n)=\dfrac{(\mu)_n}{n!}$,the following function recently studied by Kumar [56, p.13, eq. (1.9)] in the following manner:

$$\in_{\alpha,\beta}^{\mu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(\delta n+a)^s\,\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(3.2.17)$$

(where $\mathrm{Re}(\mu)>0$; $\mathrm{Re}(\delta)>0$; $\mathrm{Re}(a)>0$; $\mathrm{Re}(s)\ge 0$; $\mathrm{Re}(\alpha)>0$; $\mathrm{Re}(\beta)>0$ and $|z|\le 1$).

To give more extension in view of the parameters the more generalized function defined in (2.2.17) and given as below [43, p.954, eq. (1.17)]

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\rho n}(a+n)^{s}\Gamma(\alpha n+\beta)}\frac{z^{n}}{n!} \qquad \ldots(3.2.18)$$

where ($\alpha,\beta,\mu,\lambda \in C$; $a,\sigma \in C/Z_0^{-}$; $\delta,\nu,\rho \in R^{+}$; $\rho-\delta-\nu>-1$ when $s,z\in C$; $\rho-\delta-\nu=-1$ and $s\in C$ when $|z|<\xi^{*}=\delta^{-\delta}\nu^{-\nu}\rho^{\rho}$; $\rho-\delta-\nu=-1$ and $\mathrm{Re}(s+\sigma-\mu-\lambda)>1$ when $|z|=\xi^{*}$)

The integral representation of the function defined in (3.2.18) is given in (2.4.1) and also studied by Jaimini and Tatwal in [43, p.954, eq. (1.18)] as the follows.

**Theorem-1**

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0$; $\alpha,\beta,\mu,\lambda \in C$; $a,\sigma \in C/Z_0^{-}$; $\delta,\nu,\rho \in R^{+}$; $\rho-\delta-\nu \geq -1$ with $|z|<\xi^{*}=\delta^{-\delta}\nu^{-\nu}\rho^{\rho}$

the integral representation of the function $\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)$ holds true

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\int_{0}^{\infty}e^{-at}t^{s-1}E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)dt \qquad \ldots(3.2.19)$$

where $E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittag-Leffler function defined in (3.2.13).

In this chapter, the following well known facts and rules are required.

(i) The Riemann-Liouville fractional derivatives operator $D_z^{\mu}$ is defined by [21 p.181]:

$$D_z^{\mu}\{f(z)\}=\begin{cases}\dfrac{1}{\Gamma(-\mu)}\displaystyle\int_{0}^{z}(z-t)^{-\mu-1}f(t)\,dt, & (\mathrm{Re}(\mu)<0)\\[2ex] \dfrac{d^{m}}{dz^{m}}D_z^{\mu-m}\{f(z)\}, & (m-1\leq \mathrm{Re}(\mu)<m\,(m\in N))\end{cases} \qquad \ldots(3.2.20)$$

It is known that

$$D_z^{\mu}\left\{z^{\lambda}\right\}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}z^{\lambda-\mu},\left(\operatorname{Re}(\lambda)>-1\right) \qquad \ldots(3.2.21)$$

(ii) The Gamma integral [29, p.884, eq.312(2)]

$$\frac{\Gamma(\alpha)}{a^{\alpha}}=\int_0^{\infty}x^{\alpha-1}e^{-ax}dx,\operatorname{Re}(\alpha)>0 \qquad \ldots(3.2.22)$$

(iii) The Binomial expansion [96, p.20, eq. (29)]

$$(1-x)^{-a}=\sum_{n=0}^{\infty}(a)_n\frac{x^n}{n!} \qquad \ldots(3.2.23)$$

where $(a)_n$ is the pochhammer's symbol defined in (3.2.5).

## SECTION-A

**3.3** In this section the three integral representations and one fractional derivative for general function defined in (3.2.18) are established. The general function studied in this paper unifies the Mittag-Leffler function and the Hurwitz-Lerch Zeta function. Therefore, as applications of these results we have evaluated certain type of integrals involving the Mittag-Leffler function which gives the general function as resultant. Also many known and new results related to the Hurwitz-Lerch Zeta function are shown to be derived.

## 3.4 MAIN THEOREMS RELATED TO $\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)$

We have obtained here certain integral representations and fractional derivative formulae for general function defined in (3.2.18). These formulae are given in the following four theorems.

**Theorem-1**

For w, $\lambda,\mu\in C$; $\sigma\in C/Z_0^-$; $p\in N_0$; $\min\left\{\operatorname{Re}(a),\operatorname{Re}(s)\right\}>0$; $\delta,\nu\in R^+$; $\rho-\delta-\nu\geq-1$

with equality when $|z|<\xi^{\rho}=\delta^{-\delta}\nu^{-\nu}\rho^{\rho}$,

we have the following integral representation

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{\mathrm{k}!}\int_0^{\infty}t^{s-1}e^{-(a+kp)t}\left(1-e^{-pt}\right)^{w}E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)dt \qquad \ldots(3.4.1)$$

provided that each side of (3.4.1) exists and $E^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z)$ is the Mittag-Leffler function defined in (3.2.13).

**Theorem-2**

For $\lambda,\mu\in C$; $\sigma\in C/Z_0^-$; $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0$; $\delta,\nu,\rho\in R^+$; $\rho-\delta-\nu\geq -1$ with equality when $|z|<\xi^\rho=\delta^{-\delta}\nu^{-\nu}\rho^\rho$,

we have the following integral representation

$$\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z,s,a)=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\int_0^\infty t^{s-1}e^{-\left(a+\frac{k(k+1)}{2}\right)t}\left(1-e^{-(k+1)t}\right)\ E^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}\left(ze^{-t}\right)dt \qquad \ldots(3.4.2)$$

provided that each side of (3.4.2) exists and $E^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z)$ is the Mittag-Leffler function defined in (3.2.13).

**Theorem-3**

Let $(a_n)_{n\in N_0}$ be a positive sequence such that the following Infinite series $\sum_{n=0}^{\infty}e^{-a_n t}$ converges for any $t\in R^+$ and for $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0$

we have the following integral representation

$$\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z,s,a)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^\infty t^{s-1}e^{-(a-a_0+a_n)t}\left[1-e^{-(a_{n+1}-a_n)t}\right]E^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}\left(ze^{-t}\right)dt \qquad \ldots(3.4.3)$$

provided that each side of (3.4.3) exists and $E^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z)$ is the Mittag-Leffler function defined in (3.2.13).

**Theorem-4**

For $\mathrm{Re}(v)>0$ and $\rho>0$, we have the following fractional derivative

$$D_z^{\sigma-\tau}\left\{z^{\sigma-1}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}\left(z^\rho,s,a\right)\right\}=\frac{\Gamma\sigma}{\Gamma\tau}z^{\tau-1}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\tau,\rho}\left(z^\rho,s,a\right) \qquad \ldots(3.4.4)$$

provided that each side of (3.4.4) exists.

**OUTLINE OF PROOFS**

**Proof of (3.4.1)**

To prove result (3.4.1), let the right hand side of (3.4.1) is denoted by $\Delta_1$ i.e.

$$\Delta_1 = \frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\int_0^{\infty} t^{s-1}e^{-(a+kp)t}\left(1-e^{-pt}\right)^w E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right)dt$$

On changing the order of integration and summation, we have

$$\Delta_1 = \frac{1}{\Gamma s}\int_0^{\infty}\left\{\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\left(e^{-pt}\right)^k\right\}t^{s-1}e^{-at}\left(1-e^{-pt}\right)^w E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right)dt$$

Now using the binomial expansion (3.2.23), above expression takes the following form

$$\Delta_1 = \frac{1}{\Gamma s}\int_0^{\infty} t^{s-1}e^{-at}E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right)dt$$

Now evaluate the above integral in view of (3.2.19), we arrive at once the desired result in (3.4.1).

**Proof of (3.4.2)**

To prove result (3.4.2), let the right hand side of (3.4.2) is denoted by $\Delta_2$ i.e.

$$\Delta_2 = \frac{1}{\Gamma s}\sum_{k=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a+\frac{k(k+1)}{2}\right)t}\left[1-e^{-(k+1)t}\right]E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right)dt$$

Using the integral representation (3.2.19), the above integral can be written as

$$\Delta_2 = \sum_{k=0}^{\infty}\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+\frac{k(k+1)}{2}\right)-\sum_{k=0}^{\infty}\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

Which gives

$$\Delta_2 = \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}(z;s,a)+\sum_{k=1}^{\infty}\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+\frac{k(k+1)}{2}\right)-\sum_{k=0}^{\infty}\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

$$\Delta_2 = \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}(z;s,a)+\sum_{m=0}^{\infty}\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+(m+1)\left(\frac{m}{2}+1\right)\right)-\sum_{k=0}^{\infty}\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

After simplifying, the complete expression becomes $\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}(z,s,a)$. This completes the proof of required result in (3.4.2).

**Proof of (3.4.3)**

To prove result (3.4.3), let its right hand side is denoted by $\Delta_3$ i.e.

$$\Delta_3 = \frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-(a-a_0+a_n)t}\left[1-e^{-(a_{n+1}-a_n)t}\right] E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)dt$$

Using the integral representation (3.2.19) it can be written in the following form

$$\Delta_3 = \sum_{n=0}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z,s,a-a_0+a_n\right) - \sum_{n=0}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z,s,a-a_0+a_{n+1}\right)$$

Which gives

$$\Delta_3 = \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z;s,a\right) + \sum_{n=1}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z;s,a-a_0+a_n\right) - \sum_{n=0}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z,s,a-a_0+a_{n+1}\right)$$

$$\Delta_3 = \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z;s,a\right) + \sum_{m=0}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z;s,a-a_0+a_{m+1}\right) - \sum_{n=0}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z,s,a-a_0+a_{n+1}\right)$$

After simplifying, the above complete expression becomes $\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z,s,a\right)$. This completes the proof of required result in (3.4.3).

**Proof of (3.4.4)**

To prove result (3.4.4), let its left hand side is denoted by $\Delta_4$ i.e.

$$\Delta_4 = D_z^{\sigma-\tau}\left\{z^{\sigma-1} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(z^{\rho};s,a\right)\right\}$$

Using the definition of (3.2.18) and on simplification we have the following form

$$\Delta_4 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\rho n}(a+n)^s\,\Gamma(\alpha n+\beta)\,n!} D_z^{\sigma-\tau}\left\{z^{\rho n+\sigma-1}\right\}$$

Now on using (3.2.21) we have

$$\Delta_4 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\rho n}(a+n)^s\,\Gamma(\alpha n+\beta)\,n!}\ \frac{\Gamma(\rho n+\sigma)}{\Gamma(\rho n+\tau)} z^{\rho n+\tau-1}$$

In view of (3.2.5), it can be written as follows

$$\Delta_4 = \frac{\Gamma\sigma}{\Gamma\tau} z^{\tau-1}\sum_{n=0}^{\infty} \frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\tau)_{\rho n}(a+n)^s\,\Gamma(\alpha n+\beta)} \frac{\left(z^{\rho}\right)^n}{n!}$$

On interpreting the series in view of (3.2.18) we achieve atonce the desired result in (3.4.4).

## 3.5 MILD EXTENSION AND ITS INTEGRAL REPRESENTATION

We also studied here the mild generalization of the function defined in (3.2.18) in view of parameters. It is defined in (2.6.1) and given as below [43, p.958, eq. (3.1)]:

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(a+n)^s\,\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \dots(3.5.1)$$

where $(p,q\in N_0$; $s,z,\mu_i,\alpha,\beta\in C$; $a,\sigma_j\in C/Z_0^-$; $\delta_i,\rho_j\in R^+$,

$\forall(i=1,2,.......p;\ j=1,2........q)$, and $(\mu_p)\equiv\mu_1,\mu_2,...,\mu_p$; $(\delta_p)\equiv\delta_1,\delta_2,...,\delta_p$ and similar notation stands for $(\sigma_p)$ and $(\rho_q)$

The integral representations for the function defined in (3.5.1) is given by (2.6.3) and recently studied by Jaimini and Tatwal [43, p.959, eq. (3.2)] as the following theorem.

**Theorem-1**

For $p,q\in N_0$; $s,z,\mu_i,\alpha,\beta\in C$; $a,\sigma_j\in C/Z_0^-$; $\delta_i,\rho_j\in R^+$

$,\forall(i=1,2,.......p;\ j=1,2........q)$,

we have,

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}e^{-at}t^{s-1}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt \qquad \dots(3.5.2)$$

provided that each side of (3.5.2) exists and $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is the multi-parameter Mittag-Leffler function defined by (2.6.2) in the following manner [43, p.959, eq. (3.3)]:

$$E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\quad . \qquad \dots(3.5.3)$$

## 3.6 MAIN THEOREMS RELATED TO $\in^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}(z;s,a)$

Certain new series integral representations for the generalized function defined in (3.5.1) and the fractional derivative involving this general function are established here as the following theorems.

**Theorem-1**

For all $i=1,2,.......\text{p}$; $j=1,2,.......\text{q}$; $p,q\in N_0$; $s,z,\mu_i\in C$; $a,\sigma_j\in C/Z_0^-$; $\delta_i,\rho_j\in R^+$ $\min\{\text{Re}(a),\text{Re}(s)\}>0$, we have

$$\in^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}(z;s,a)=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\int_0^{\infty}t^{s-1}e^{-(a+kp)t}\left(1-e^{-pt}\right)^w E^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}\left(ze^{-t}\right)dt \quad \ldots(3.6.1)$$

provided that each side of (3.6.1) exists and $E^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}(z)$ is the multi-parameter Mittag-Leffler function defined in (3.5.3).

**Theorem-2**

For all $i=1,2,.......\text{p}$; $j=1,2,.......\text{q}$; $p,q\in N_0$; $s,z,\mu_i\in C$; $a,\sigma_j\in C/Z_0^-$; $\delta_i,\rho_j\in R^+$ $\min\{\text{Re}(a),\text{Re}(s)\}>0$, we have

$$\in^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}(z;s,a)=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\int_0^{\infty}t^{s-1}e^{-\left(a+\frac{k(k+1)}{2}\right)t}\left[1-e^{-(k+1)t}\right]E^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}\left(ze^{-t}\right)dt \quad \ldots(3.6.2)$$

provided that each side of (3.6.2) exists and $E^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}(z)$ is the multi-parameter Mittag-Leffler function defined in (3.5.3).

**Theorem-3**

For all $i=1,2,.......\text{p}$; $j=1,2,.......\text{q}$; $p,q\in N_0$; $s,z,\mu_i\in C$; $a,\sigma_j\in C/Z_0^-$; $\delta_i,\rho_j\in R^+$ $\min\{\text{Re}(a),\text{Re}(s)\}>0$. Also let $(a_n)_{n\in N_0}$ be the positive sequence such that the following infinite series $\sum_{n=0}^{\infty}e^{-a_nt}$ converges for any $t\in R^+$ then

$$\in^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}(z;s,a)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty}t^{s-1}e^{-(a-a_0+a_n)t}\left[1-e^{-(a_{n+1}-a_n)t}\right]E^{(\mu_p),(\delta_p)}_{\alpha,\beta;(\sigma_q),(\rho_q)}\left(ze^{-t}\right)dt$$

$\ldots(3.6.3)$

provided that each side of (3.6.3) exists and $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is the multi-parameter Mittag-Leffler function defined in (3.5.3).

**Theorem-4**

For $\mathrm{Re}(v)>0;\lambda>0$ we have the following fractional derivative

$$D_z^{v-\tau}\left\{z^{v-1}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_q)}\left(z^{\lambda};s,a\right)\right\}=\frac{\Gamma v}{\Gamma \tau}z^{\tau-1}\in_{\alpha,\beta;(\sigma_q),\tau,(\rho_q),\lambda}^{(\mu_p),v;(\delta_q),\lambda}\left(z^{\lambda};s,a\right) \qquad \ldots(3.6.4)$$

provided that each side of (3.6.4) exists.

**OUTLINE OF PROOFS**

**Proof of (3.6.1)**

To prove result (3.6.1), let the right hand side of (3.6.1) is denoted by $\Delta_1$ i.e.

$$\Delta_1=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\int_0^{\infty}t^{s-1}e^{-(a+kp)t}\left(1-e^{-pt}\right)^{w}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt$$

On changing the order of integration and summation, we have

$$\Delta_1=\frac{1}{\Gamma s}\int_0^{\infty}\left\{\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\left(e^{-pt}\right)^{k}\right\}t^{s-1}e^{-at}\left(1-e^{-pt}\right)^{w}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt$$

Now using the binomial expansion (3.2.23), above expression takes the following form

$$\Delta_1=\frac{1}{\Gamma s}\int_0^{\infty}t^{s-1}e^{-at}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt$$

Now evaluate the above integral in view of (3.5.2), we arrive atonce the desired result in (3.6.1).

**Proof of (3.6.2)**

To prove result (3.6.2), let the right hand side of (3.6.2) is denoted by $\Delta_2$ i.e.

$$\Delta_2=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\int_0^{\infty}t^{s-1}e^{-\left(a+\frac{k(k+1)}{2}\right)t}\left[1-e^{-(k+1)t}\right]E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt$$

Using the integral representation (3.5.2), the above integral can be written as

$$\Delta_2=\sum_{k=0}^{\infty}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z;s,a+\frac{k(k+1)}{2}\right)-\sum_{k=0}^{\infty}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

Which gives

$$\Delta_2 = \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a) + \sum_{k=1}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z;s,a+\frac{k(k+1)}{2}\right)$$

$$-\sum_{k=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

$$\Delta_2 = \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a) + \sum_{m=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z;s,a+(m+1)\left(\frac{m}{2}+1\right)\right)$$

$$-\sum_{k=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

After simplifying, the complete expression it becomes $\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a)$. This complete the proof the required result in (3.6.2).

**Proof of (3.6.3)**

To prove result (3.6.3), let its right hand side is denoted by $\Delta_3$ i.e.

$$\Delta_3 = \frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-(a-a_0+a_n)t}\left[1-e^{-(a_{n+1}-a_n)t}\right] E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)\, dt$$

Using the integral representation (3.5.2) it can be written in the following form

$$\Delta_3 = \sum_{n=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a-a_0+a_n) - \sum_{n=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a-a_0+a_{n+1})$$

Which gives

$$\Delta_3 = \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a) + \sum_{n=1}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a-a_0+a_n)$$

$$-\sum_{n=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a-a_0+a_{n+1})$$

$$\Delta_3 = \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a) + \sum_{m=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a-a_0+a_{m+1})$$

$$-\sum_{n=0}^{\infty} \in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a-a_0+a_{n+1})$$

After simplifying, the above complete expression becomes $\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a)$. This completes the proof of the required result in (3.6.3).

**Proof of (3.6.4)**

To prove result (3.6.4), let its left hand side is denoted by $\Delta_4$ i.e.

$$\Delta_4 = D_z^{\nu-\tau}\left\{z^{\nu-1}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_q)}\left(z^{\lambda};s,a\right)\right\}$$

Using the definition of (3.5.1) and on simplification we have the following form

$$\Delta_4 = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(a+n)^s\,\Gamma(\alpha n+\beta)\,n!}D_z^{\nu-\tau}\left\{z^{\lambda n+\nu-1}\right\}$$

Now on using (3.2.21) we have

$$\Delta_4 = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(a+n)^s\,\Gamma(\alpha n+\beta)\,n!}\frac{\Gamma(\lambda n+\nu)}{\Gamma(\lambda n+\tau)}z^{\lambda n+\tau-1}$$

In view of (3.2.5), it can be written as follows

$$\Delta_4 = \frac{\Gamma\nu}{\Gamma\tau}z^{\tau-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}(\nu)_{\lambda n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(\tau)_{\lambda n}(a+n)^s\,\Gamma(\alpha n+\beta)}\frac{\left(z^{\lambda}\right)^n}{n!}$$

On interpreting the series in view of (3.5.1) we achieve atonce the desired result in (3.6.4).

## 3.7 SPECIAL CASES

(i) For limit $\alpha \to 0$ the results (3.2.19), (3.4.3), (3.4.4), (3.6.3), (3.6.4) reduce respectively to the known results due to Srivastava et al.[101, p.494, eq.(2.4); p.496, eq.(2.18); p.502, eq.(5.1); p.505, eq.(6.9); p.505, eq.(6.8)].

(ii) If in results (3.4.1) and (3.4.2), on letting limit $\alpha \to 0$ these reduce to the known results due to Gupta ([30], pp. 188-189, eqs. (3.9.1), (3.9.2) at b=0).

(iii) If we take limit $\alpha \to 0$ and $\delta = v = \rho = 1$ in results (3.4.1) and (3.4.2) then these reduce to the known results due to Garg et al [23, p. 314, eqs. (2.4), (2.3)] which in turn at $\mu = \sigma, \lambda = 1$ gives the results [60, p.129, eqs. (46), (47)].

(iv) If we consider limit $\alpha \to 0$ and $\delta = v = \rho = 1$ , $\lambda = \sigma$ in result (3.4.4) then it reduce to the known result due to Gupta [30, p.149, eq.(3.4.7)].

(v) If we take limit $\alpha \to 0$ and $\delta = v = \rho = 1, \tau = \mu$ in result (3.4.4) then it reduce to the known result due to Gupta [30, p.149, eq. (3.4.8)].

(vi) For limit $\alpha \to 0$, the result (3.6.4) reduce to the known result due to Gupta ([30], p.175, eq.(3.8.16) at $b = 0$ and $\beta = 1$).

## SECTION-B

**3.8** Motivated by the work of Srivastava [94] and Kumar [56], and to extend the work of Section-A of this chapter here in this section we consider the following multivariable analogue of the general function defined in (3.2.18) which is defined and represented in the following manner:

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right) = \sum_{n_1,...,n_r=0}^{\infty} \frac{1}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^s \Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)} \prod_{i=1}^{r}\left[\frac{\left(\mu_i\right)_{\delta_i n_i} z_i^{n_i}}{n_i\,!}\right] \quad ...(3.8.1)$$

where $\forall\ i = 1,2,...,r;\ \mu_i, \alpha_i, s, z \in C;\ \delta_i \in R^+;\ \min\left\{\mathrm{Re}(a), \mathrm{Re}(s)\right\} > 0;\ \max_{1\le i\le r}|z_i| < 1,$

$\mathrm{Re}(\alpha_i) > 0;\ \mathrm{Re}(p_i) > 0.$

In this section, we obtained the integral representation of the general function defined in (3.8.1). Three integral expressions are established for this multivariable general function. The mild extension of these results is also established here. As application of these results the integral expressions for simpler function related to Hurwitz-Lerch Zeta function are shown to be derived. Due to generalization of parameters and variables the main results established in this section also provide the extensions of the results studied in earlier Section-A of this chapter.

## 3.9 DEFINITIONS REQUIRED

The multivariable extension of the function (3.2.6) is defined and studied by Raina and Chhajed [80, p.98, eq.(5.1)]. We define this function at $b=0$ in the following manner:

$$\phi_{(\mu_i)}^{(p_i)}\left(x_1,...,x_n;\alpha,a\right)=\sum_{m_1,...,m_n=0}^{\infty}\left(a+\Omega\right)^{-\alpha}\prod_{i=1}^{n}\left[\frac{\left(\mu_i\right)_{m_i}x_i^{m_i}}{\left(m_i\right)}\right] \qquad \ldots(3.9.1)$$

where $\mu\geq 1\ \left(i=1,2,...,\mathrm{n}\right);\ \mathrm{Re}\left(\alpha\right)>0;\ \mathrm{Re}\left(p_i\right)>0;\ \max_{1\leq i\leq n}\left|x_i\right|<1;\ \Omega=\sum_{i=1}^{n}p_i m_i$

A multivariable extension of Mittag-Leffler function defined in (3.2.10) is studied by Gautam [25] and Saxena et.al [88, p.536, eq. (1.14)], it is defined and represented as follows:

$$E_{(\rho_r),\lambda}^{(\gamma_r)}\left(z_1,...,z_r\right)=\sum_{k_1,...,k_r=0}^{\infty}\frac{\left(\gamma_1\right)_{k_1}\cdots\left(\gamma_r\right)_{k_r}}{\Gamma\left(\lambda+k_1\rho_1+...+k_r\rho_r\right)}\frac{z_1^{k_1}...z_r^{k_r}}{k_1!,...,k_r!} \qquad \ldots(3.9.2)$$

where $\lambda,\gamma_j,\rho_j\in C;\ \mathrm{Re}\left(\rho_j\right)>0;\ j=1,2,...,r$

A mild generalization of multivariable analogue of Mittag-Leffler function (3.9.2) is also due to Saxena et.al. [88, p.547, eq. (7.1)] (see also [25],[30]).

$$E_{(\rho_r),\lambda}^{(\gamma_r);(l_r)}\left(z_1,...,z_r\right)=\sum_{k_1,...,k_r=0}^{\infty}\frac{\left(\gamma_1\right)_{k_1l_1}\cdots\left(\gamma_r\right)_{k_rl_r}}{\Gamma\left(\lambda+k_1\rho_1+...+k_r\rho_r\right)}\frac{z_1^{k_1}...z_r^{k_r}}{k_1!...k_r!} \qquad \ldots(3.9.3)$$

where $\lambda,\gamma_j,l_j,\rho_j\in C;;\mathrm{Re}\left(\rho_j\right)>0;\ \mathrm{Re}\left(l_j\right)>0;\ \lambda\notin Z_0^-=\left\{0,-1,-2,...\right\};\ j=1,2,...,r$

The integral representation of the function (3.8.1) is obtained here.

**Integral Representation**

For all $i=1\ldots\mathrm{r};\ \delta_i\in R^+\ \mu_i,\alpha_i,\mathrm{s},\mathrm{z}\in C;\ min\left\{\mathrm{Re}\left(a\right),\mathrm{Re(s)}\right\}>0;\ \mathrm{Re}\left(p_i\right)>0;$

$\mathrm{Re}\left(\alpha_i\right)>0;$ and $\max_{1\leq i\leq r}\left|z_i\right|<1,$

we have the following integral representation of the function defined in (3.8.1)

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1e^{-p_1t},...,z_re^{-p_rt}\right)dt \qquad \ldots(3.9.4)$$

where $E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function defined in (3.9.3).

**Proof of (3.9.4)**

We have the multivariable general function (3.8.1) in the following form,

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{n_1,...,n_r=0}^{\infty}\left\{\frac{\Gamma s}{\left(a+\sum_{i=1}^{r}p_in_i\right)^s}\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\prod_{i=1}^{r}\left\{\frac{\left\{\left(\mu_i\right)_{\delta_in_i}z_i^{n_i}\right\}}{n_i!}\right\}$$

In view of (3.2.22) we have

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{n_1,...,n_r=0}^{\infty}\left\{\int_0^\infty t^{s-1}e^{-\left(a+\sum_{i=1}^{r}p_in_i\right)t}dt\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\prod_{i=1}^{r}\left\{\frac{\left\{\left(\mu_i\right)_{\delta_in_i}z_i^{n_i}\right\}}{n_i!}\right\}$$

on interchanging the order of integration and summation it takes the following form

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\prod_{i=1}^{r}\left\{\frac{\left(\mu_i\right)_{\delta_in_i}\left(z_i^{n_i}e^{-p_it}\right)^{n_i}}{n_i!}\right\}dt$$

on interpreting the inner summation in view of (3.9.3), we atonce arrive at the required result in (3.9.4).

## 3.10 INTEGRAL EXPRESSIONS RELATED TO $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)$

In this section, we establish the following three integral expressions associated to multivariable general function defined in (3.8.1).

**Theorem-1**

For all $i=1...r$; $\delta_i\in R^+$ $\mu_i,\alpha_i,s,z\in C$; $min\left\{\mathrm{Re}\left(a\right),\mathrm{Re(s)}\right\}>0$; $\mathrm{Re}\left(p_i\right)>0$;

$\mathrm{Re}\left(\alpha_i\right)>0$; and $\max_{1\le i\le r}\left|z_i\right|<1$,

we have the following integral expression

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{l=0}^{\infty}\frac{(\gamma)_l}{l!}\int_0^{\infty}t^{s-1}e^{-(a+lp)t}\left(1-e^{-pt}\right)^{\gamma}E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1e^{-p_1t},...,z_re^{-p_rt}\right)dt$$

…(3.10.1)

provided that each the side of (3.10.1) exists and $E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function defined in (3.9.3).

**Theorem-2**

For all $i=1...\text{r};\ \delta_i\in R^{+}\ \mu_i,\alpha_i,\text{s},\text{z}\in C;\ min\{\text{Re}(a),\text{Re(s)}\}>0;\ \text{Re}(p_i)>0;$ $\text{Re}(\alpha_i)>0;$ and $\max_{1\le i\le r}|z_i|<1,$

we have the following integral expression

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{l=0}^{\infty}\int_0^{\infty}t^{s-1}e^{-\left(a+\frac{l(l+1)}{2}\right)t}\left(1-e^{-(l+1)t}\right)E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1e^{-p_1t},...,z_re^{-p_rt}\right)dt$$

…(3.10.2)

provided that each the side of (3.10.2) exists and $E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function defined in (3.9.3).

**Theorem-3**

Let $\left(\gamma_n\right)_{n\in N_0}$ be a positive sequence such that the following infinite $\sum_{n=0}^{\infty}e^{-\gamma_n t}$ converges for any $t\in R^{+}$ and for all $i=1...\text{r};\ \delta_i\in R^{+}\ \mu_i,\alpha_i,\text{s},\text{z}\in C;\ min\{\text{Re}(a),\text{Re(s)}\}>0;$ $\text{Re}(p_i)>0;\ \text{Re}(\alpha_i)>0;$ and $\max_{1\le i\le r}(|z_i|)<1$ then,

we have the following integral expression

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty}t^{s-1}e^{-(a-\gamma_0+\gamma_n)t}\left[1-e^{-(\gamma_{n+1}-\gamma_n)t}\right]E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1e^{-p_1t},...,z_re^{-p_rt}\right)dt$$

…(3.10.3)

provided that each the side of (3.10.3) exists and $E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function defined in (3.9.3).

**OUTLINE OF PROOFS**

**Proof of (3.10.1)**

Let the right hand side of (3.10.1) is denoted by $\Delta_1$ i.e.

$$\Delta_1 = \frac{1}{\Gamma s} \sum_{l=0}^{\infty} \frac{(\gamma)_l}{l!} \int_0^{\infty} t^{s-1} e^{-(a+lp)t} \left(1-e^{-pt}\right)^{\gamma} E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t}, ..., z_r e^{-p_r t}\right) dt,$$

On interchanging the order of integration and summation, we have the following form

$$\Delta_1 = \frac{1}{\Gamma s} \int_0^{\infty} \left\{ \sum_{l=0}^{\infty} \frac{(\gamma)_l}{l!} \left(e^{-pt}\right)^l \right\} t^{s-1} e^{-at} \left(1-e^{-pt}\right)^{\gamma} E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t}, ..., z_r e^{-p_r t}\right) dt,$$

Now in view of (3.2.23), we have

$$\Delta_1 = \frac{1}{\Gamma s} \int_0^{\infty} t^{s-1} e^{-at} E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t}, ..., z_r e^{-p_r t}\right) dt,$$

On evaluating the above integral with the help of (3.9.4), we atonce arrive at the desired result in (3.10.1).

**Proof of (3.10.2)**

Let the right hand side of (3.10.2) is denoted by $\Delta_2$ i.e.

$$\Delta_2 = \frac{1}{\Gamma s} \sum_{l=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-\left(a+\frac{l(l+1)}{2}\right)t} \left[1-e^{-(l+1)t}\right] E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t}, ..., z_r e^{-p_r t}\right) dt,$$

it can be written straightly in the following form

$$\Delta_2 = \frac{1}{\Gamma s} \sum_{l=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-\left(a+\frac{l(l+1)}{2}\right)t} E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t}, ..., z_r e^{-p_r t}\right) dt$$

$$- \frac{1}{\Gamma s} \sum_{l=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-\left(a+(l+1)\left(\frac{l}{2}+1\right)\right)t} E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t}, ..., z_r e^{-p_r t}\right) dt$$

On evaluating the above integrals with the help of (3.9.4) it can be written in the following form

$$\Delta_2 = \sum_{l=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[z_1, ..., z_r; s, a + \frac{l(l+1)}{2}\right] - \sum_{l=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[z_1, ..., z_r; s, a + (l+1)\left(\frac{l}{2}+1\right)\right],$$

which gives

$$\Delta_2 = \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(z_1, ..., z_r; s, a\right) + \sum_{l=1}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[z_1, ..., z_r; s, a + \frac{l(l+1)}{2}\right]$$

$$-\sum_{l=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[ z_1,...,z_r; s, a+(l+1)\left(\frac{l}{2}+1\right)\right]$$

It can be written as

$$\Delta_2 = \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(z_1,...,z_r; s, a\right) + \sum_{m=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[ z_1,...,z_r; s, a+(m+1)\left(\frac{m}{2}+1\right)\right]$$

$$-\sum_{l=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[ z_1,...,z_r; s, a+(l+1)\left(\frac{l}{2}+1\right)\right]$$

on simplifying, the above complete expression becomes $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(z_1...z_r; s, a\right)$. This completes the proof of the result in (3.10.2).

**Proof of (3.10.3)**

Let the right hand side of (3.10.3) is denoted by $\Delta_3$ i.e.

$$\Delta_3 = \frac{1}{\Gamma s} \sum_{n=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-(a-\gamma_0+\gamma_n)t} \left[1-e^{-(\gamma_{n+1}-\gamma_n)t}\right] E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t},..., z_r e^{-p_r t}\right) dt,$$

it can be written in the following form

$$\Delta_3 = \frac{1}{\Gamma s} \sum_{n=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-(a-\gamma_0+\gamma_n)t} E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t},..., z_r e^{-p_r t}\right) dt,$$

$$-\frac{1}{\Gamma s} \sum_{n=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-(a+\gamma_{n+1}-\gamma_0)t} E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)} \left(z_1 e^{-p_1 t},..., z_r e^{-p_r t}\right) dt$$

On evaluating the above integrals with the help of (3.9.4), it takes the following form

$$\Delta_3 = \sum_{n=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[z_1,...,z_r; s, \left(a-\gamma_0+\gamma_n\right)\right] - \sum_{n=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[z_1,...,z_r; s, \left(a+\gamma_{n+1}-\gamma_0\right)\right]$$

which gives

$$\Delta_3 = \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(z_1,...,z_r; s, a\right) + \sum_{n=1}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[z_1,...,z_r; s, \left(a-\gamma_0+\gamma_n\right)\right]$$

$$-\sum_{n=0}^{\infty} \in_{(\alpha_r),\beta;p_i}^{(\mu_r)(\delta_r)} \left[z_1,...,z_r; s, \left(a+\gamma_{n+1}-\gamma_0\right)\right]$$

It can be written as

$$\Delta_3 = \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(z_1,...,z_r; s, a\right) + \sum_{m=0}^{\infty} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left[z_1,...,z_r; s, \left(a-\gamma_0+\gamma_{m+1}\right)\right]$$

$$-\sum_{n=0}^{\infty} \in_{(\alpha_r),\beta;p_i}^{(\mu_r)(\delta_r)} \left[z_1,...,z_r; s, \left(a+\gamma_{n+1}-\gamma_0\right)\right]$$

on simplifying, the above complete expression becomes $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1...z_r;s,a\right)$. This completes the proof of result in (3.10.3).

## 3.11 MILD EXTENSION OF MULTIVARIABLE GENERAL FUNCTION

A mild extension of general function defined in (3.8.1), in view of the parameters is studied here. The extended general function is defined and represented in the following manner:

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a\right)$$

$$=\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i!}\right] \qquad ...(3.11.1)$$

where $\forall\ i=1...r;\ p_i,q_i\in N_0;\ \delta_j^{(i)},\rho_k^{(i)}\in R^+;\ a,\sigma_k^{(i)}\in C/Z_0^-;\gamma_i,s,z,\mu_j^{(i)},\alpha_i,\beta\in C$ $\left(j=1...p_i;k=1...q_i\right)$, $\max\limits_{1\le i\le r}|z_i|<1$, and for all $i=1...r$, the notation $(\mu_j^{(i)})\equiv\mu_1^{(i)},...,\mu_{p_i}^{(i)}$ and $(\delta_j^{(i)})$ abbreviate the same array of parameters as in $(\mu_j^{(i)})$. $(\sigma_k^{(i)})\equiv\sigma_1^{(i)},...,\sigma_{q_i}^{(i)}$ and $(\rho_k^{(i)})$ abbreviate the same array of parameters as in $(\sigma_k^{(i)})$ and $(\alpha_i)\equiv\alpha_1,...,\alpha_r;(\gamma_i)\equiv\gamma_1,...,\gamma_r$.

## 3.12 DEFINATIONS REQUIRED

We consider here the following extended Mittag-Leffler multivariable function. It is represented and defined in the following manner:

$$E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r\right)=\sum_{n_1,..,n_r=0}^{\infty}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i!}\right] \qquad ...(3.12.1)$$

$\forall(i=1,...,r;\ j=1,...,p_i;k=1,...,q)$, $\mu_j^{(i)},\delta_j^{(i)},\sigma_k^{(i)},\rho_k^{(i)},\alpha_i,\beta\in C,\ p_i,q_i\in N_0$ and

$\min\left\{\mathrm{Re}(\mu_j^{(i)}),\mathrm{Re}(\alpha_i),\mathrm{Re}(\beta),\mathrm{Re}(\mu_j^{(i)}),\mathrm{Re}(\delta_j^{(i)}),\mathrm{Re}(\sigma_k^{(i)}),\mathrm{Re}(\rho_k^{(i)})\right\}>0$.

the notation $(\mu_j^{(i)}) \equiv \mu_1^{(i)},...,\mu_{p_i}^{(i)}$ and $(\delta_j^{(i)})$ abbreviate the same array of parameters as in $(\mu_j^{(i)})$. $(\sigma_k^{(i)}) \equiv \sigma_1^{(i)},...,\sigma_{q_i}^{(i)}$ and $(\rho_k^{(i)})$ abbreviate the same array of parameters as in $(\sigma_k^{(i)})$ and $(\alpha_i) \equiv \alpha_1,...,\alpha_r$.

**Remark:** At $p_i = q_i = 1$ $(\forall i = 1,.....r)$ the extended multivariable function (3.12.1) reduce to the multivariable Mittag-Leffler function due to Gurjar [33, p.7, eq. (1.8)].

## 3.13 INTEGRAL EXPRESSIONS RELATED TO $\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a\right)$

Certain new series integral expressions for the mild extension of the function defined in (3.11.1) are established here as the following theorems.

**Theorem-1**

For all $i = 1...r$; $j = 1...p_i$; $k = 1...q_i$; $p_i, q_i \in N_0$; $\delta_j^{(i)}, \rho_k^{(i)} \in R^+$; $a, \sigma_k^{(i)} \in C/Z_0^-$ $\gamma_i, s, z, \mu_j^{(i)}, \alpha_i, \beta \in C$, $\max\limits_{1\le i\le r}|z_i| < 1$,

we have the following integral representation of the function defined in (3.11.1)

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a\right) = \frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_re^{-\gamma_r t}\right)dt$$

...(3.13.1)

where $E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r\right)$ is the extended multivariable Mittag-Leffler function defined in (3.12.1).

**Theorem-2**

For all $\left(i = 1...r;\ j = 1...p_i;\ k = 1...q_i\right)$, $p_i, q_i \in N_0$; $\delta_j^{(i)}, \rho_k^{(i)} \in R^+$; $a, \sigma_k^{(i)} \in C/Z_0^-$ $\gamma_i, s, z, \mu_j^{(i)}, \alpha_i, \beta \in C$, $\max\limits_{1\le i\le r}|z_i| < 1$,

we have the following integral expression

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a\right)$$

$$=\frac{1}{\Gamma s}\sum_{l=0}^{\infty}\frac{(\omega)_l}{l!}\int_0^{\infty} t^{s-1}e^{-(a+lp)t}\left(1-e^{-pt}\right)^{\omega} E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right)\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt \qquad ...(3.13.2)$$

provided that each the side of (3.13.2) exists and $E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function defined in (3.12.1).

**Theorem-3**

For all $\left(i=1...\text{r};\ j=1...p_i;\ \text{k}=1...\text{q}_i\right),\ p_i,\text{q}_i\in N_0;\ \delta_j^{(i)},\rho_k^{(i)}\in R^+;\ a,\sigma_k^{(i)}\in C/Z_0^-$

$\gamma_i, s, \text{z}, \mu_j^{(i)}, \alpha_i, \beta\in C,\ \max\limits_{1\le i\le r}|z_i|<1,$

we have the following integral expression

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,\text{z}_r;s,a\right)$$

$$=\frac{1}{\Gamma s}\sum_{l=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a+\frac{l(l+1)}{2}\right)t}\left[1-e^{-(l+1)t}\right]E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt \qquad ...(3.13.3)$$

provide that each the side of (3.13.3) exists and $E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function defined in (3.12.1).

**Theorem-4**

For all $\left(i=1...\text{r};\ j=1...p_i;\ \text{k}=1...\text{q}_i\right);\ p_i,\text{q}_i\in N_0;\ \delta_j^{(i)},\rho_j^{(i)}\in R^+;a,\sigma_k^{(i)}\in C/Z_0^-$

$\mu_j^{(i)},\text{s},\text{z},\alpha_i,\beta_i,\gamma_i\in C;\ min\left\{\text{Re}(a),\text{Re(s)}\right\}>0;\ \text{Re}\left(p_i\right)>0;\ \text{Re}\left(\alpha_i\right)>0;\ \max\limits_{1\le i\le r}\left(|z_i|\right)<1$

and

Also let $\left(v_n\right)_{n\in N_0}$ be a positive sequence such that the following infinite $\sum\limits_{n=0}^{\infty}e^{-v_n t}$ converges for any $t\in R^+$ then,

we have the following integral expression

$$\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,\text{z}_r;s,a\right)$$

$$=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-(a-v_0+v_n)t}\left[1-e^{-(v_{n+1}-v_n)t}\right]E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt \qquad ...(3.13.4)$$

provided that each side of (3.13.4) exists and $E^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1,...,z_r\right)$ is the multivariable Mittag-Leffler function defined in (3.12.1).

**OUTLINE OF PROOFS**

**Proof of (3.13.1)**

We have the multivariable general function (3.11.1) in the following form,

$$\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{n_1,...,n_r=0}^{\infty}\left\{\frac{\Gamma s}{\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)^s}\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}$$

$$\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i\,!}\right]$$

In view of (3.2.22) we have

$$\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{n_1,...,n_r=0}^{\infty}\left\{\int_0^{\infty}t^{s-1}e^{-\left(a+\sum_{i=1}^{r}\gamma_i n_i\right)t}dt\right\}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}$$

$$\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i\,!}\right]$$

on interchanging the order of integration and summation it takes the following form

$$\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1,...,z_r;s,a\right)=\frac{1}{\Gamma s}\int_0^{\infty}t^{s-1}e^{-at}\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}$$

$$\prod_{i=1}^{r}\left[\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\rho_k^{(i)}n_i}}\frac{\left(z_i^{n_i}e^{-\gamma_i t}\right)^{n_i}}{n_i\,!}\right]$$

on interpreting the inner summation in view of (3.12.1), we atonce arrive at the required result in (3.13.1).

**Proof of (3.13.2)**

Let the right hand side of (3.13.2) is denoted by $\Delta_1$ i.e.

$$\Delta_1 = \frac{1}{\Gamma s}\sum_{l=0}^{\infty}\frac{(\omega)_l}{l!}\int_0^{\infty} t^{s-1}e^{-(a+lp)t}\left(1-e^{-pt}\right)^{\omega} E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right)\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt,$$

On interchanging the order of integration and summation, we have the following form

$$\Delta_1 = \frac{1}{\Gamma s}\int_0^{\infty}\left\{\sum_{l=0}^{\infty}\frac{(\omega)_l}{l!}\left(e^{-pt}\right)^l\right\} t^{s-1}e^{-at}\left(1-e^{-pt}\right)^{\omega} E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right)\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt,$$

Now in view of (3.2.23), we have

$$\Delta_1 = \frac{1}{\Gamma s}\int_0^{\infty} t^{s-1}e^{-at} E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right)\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt,$$

On evaluating the above integral with the help of (3.13.1), we atonce arrive at the desired result in (3.13.2).

**Proof of (3.13.3)**

Let the right hand side of (3.13.3) is denoted by $\Delta_2$ i.e.

$$\Delta_2 = \frac{1}{\Gamma s}\sum_{l=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a+\frac{l(l+1)}{2}\right)t}\left[1-e^{-(l+1)t}\right] E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt,$$

it can be written straightly in the following form

$$\Delta_2 = \frac{1}{\Gamma s}\sum_{l=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a+\frac{l(l+1)}{2}\right)t} E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt$$

$$-\frac{1}{\Gamma s}\sum_{l=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a+(l+1)\left(\frac{l}{2}+1\right)\right)t} E_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1e^{-\gamma_1 t},...,z_r e^{-\gamma_r t}\right) dt$$

On evaluating the above integrals with the help of (3.13.1) it can be written in the following form

$$\Delta_2 = \sum_{l=0}^{\infty} \in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left[z_1,...,z_r;s,a+\frac{l(l+1)}{2}\right]$$

$$-\sum_{l=0}^{\infty} \in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left[z_1,...,z_r;s,a+(l+1)\left(\frac{l}{2}+1\right)\right]$$

which gives

$$\Delta_2 = \in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1,\dots,z_r;s,a\right)+\sum_{l=1}^{\infty}\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left[z_1,\dots,z_r;s,a+\frac{l\left(l+1\right)}{2}\right]$$

$$-\sum_{l=0}^{\infty}\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left[z_1,\dots,z_r;s,a+\left(l+1\right)\left(\frac{l}{2}+1\right)\right]$$

It can be written as

$$\Delta_2 = \in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1,\dots,z_r;s,a\right)+\sum_{m=0}^{\infty}\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left[z_1,\dots,z_r;s,a+\left(m+1\right)\left(\frac{m}{2}+1\right)\right]$$

$$-\sum_{l=0}^{\infty}\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left[z_1,\dots,z_r;s,a+\left(l+1\right)\left(\frac{l}{2}+1\right)\right]$$

on simplifying, we get the required result given in (3.13.3).

**Proof of (3.13.4)**

Let the right hand side of (3.13.4) is denoted by $\Delta_3$ i.e.

$$\Delta_3 = \frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a-\nu_0+\nu_n\right)t}\left[1-e^{-\left(\nu_{n+1}-\nu_n\right)t}\right]E^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1e^{-\gamma_1 t},\dots,z_re^{-\gamma_r t}\right)\,dt\,,$$

it can be written in the following form

$$\Delta_3 = \frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a-\nu_0+\nu_n\right)t}E^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1e^{-\gamma_1 t},\dots,z_re^{-\gamma_r t}\right)\,dt\,,$$

$$-\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-\left(a+\nu_{n+1}-\nu_0\right)t}E^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1e^{-\gamma_1 t},\dots,z_re^{-\gamma_r t}\right)\,dt$$

On evaluating the above integrals with the help of (3.13.1), it takes the following form

$$\Delta_3 = \sum_{n=0}^{\infty}\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left[z_1,\dots,z_r;s,\left(a-\nu_0+\nu_n\right)\right]$$

$$-\sum_{n=0}^{\infty}\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left[z_1,\dots,z_r;s,\left(a+\nu_{n+1}-\nu_0\right)\right]$$

which gives

$$\Delta_3 = \in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left(z_1,\dots,z_r;s,a\right)+\sum_{n=1}^{\infty}\in^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}\left[z_1,\dots,z_r;s,\left(a-\nu_0+\nu_n\right)\right]$$

$$-\sum_{n=0}^{\infty}\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left[z_1,...,z_r;s,\left(a+\nu_{n+1}-\nu_0\right)\right]$$

It can be written as

$$\Delta_3=\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,...,z_r;s,a\right)+\sum_{m=0}^{\infty}\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left[z_1,...,z_r;s,\left(a-\nu_0+\nu_{m+1}\right)\right]$$

$$-\sum_{n=0}^{\infty}\in_{(\alpha_i),\beta;(\gamma_i),\left(\sigma_k^{(i)}\right);\left(\rho_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left[z_1,...,z_r;s,\left(a-\nu_0+\nu_{n+1}\right)\right]$$

on simplifying, we get the required result given in (3.13.4).

## 3.14 APPLICATIONS

The multivariable general function defined in (3.8.1) and its mild extension defined in (3.11.1) unify all the functions associated to Hurwitz-Lerch Zeta function of one and several variables. Therefore, as the applications, many new and known results can be obtained from our main results studied in this Section-B. Some of these have been shown to derive as follows.

(i) If in (3.9.4), we apply $\mathrm{Lim}\,\alpha_i\to 0$ and take $\delta_i=1$ then it reduces to the known result due to Raina and Chhajed ([80],p.99,eq.(5.2) at $b=0$) which in turn at $r=p_1=1$ provides the known result due to Goyal and Laddha [28, p.100, eq.(1.6)].

(ii) If in theorem in (3.10.1),(3.10.2) and (3.10.3), we apply $\mathrm{Lim}\,\alpha_i\to 0$ and take $\delta_i=1$ then these results provide the following corresponding new integral expressions for multivariable Hurwitz-Lerch Zeta function $\phi_{(\mu_r)}^{(p_r)}\left(z_1...z_r;s,a\right)$ defined by Raina and Chhajed ([80], p.98,eq.(5.1) at $b=0$).

**Corrollary-1**

For all $i=1...r$; $\mu_i,s,z\in C$; $min\left\{\mathrm{Re}(a),\mathrm{Re}(s)\right\}>0$; $\mathrm{Re}\left(p_i\right)>0$; and $\max\limits_{1\le i\le r}\left(|z_i|\right)<1$

we have the following integral expression

$$\phi_{(\mu_r)}^{(p_r)}\left(z_1...z_r;s,a\right)=\frac{1}{\Gamma s}\sum_{l=0}^{\infty}\frac{(\gamma)_l}{l!}\int_0^{\infty}t^{s-1}e^{-(a+lp)t}\left(1-e^{-pt}\right)^{\gamma}\prod_{i=1}^{r}\left(1-z_ie^{-p_it}\right)^{-\mu_i}dt \qquad ...(3.14.1)$$

**Corrollary-2**

For all $i=1\ldots r$; $\mu_i, s, z \in C$; $min\{\mathrm{Re}(a), \mathrm{Re}(s)\} > 0$; $\mathrm{Re}(p_i) > 0$; and

$\max_{1\le i\le r}(|z_i|) < 1$

we have the following integral expression

$$\phi_{(\mu_r)}^{(p_r)}(z_1 \ldots z_r; s, a) = \frac{1}{\Gamma s}\sum_{l=0}^{\infty}\int_0^{\infty} t^{s-1} e^{-\left(a+\frac{l(l+1)}{2}\right)t}\left[1-e^{-(l+1)t}\right]\prod_{i=1}^{r}\left(1-z_i e^{-p_i t}\right)^{-\mu_i} dt$$

…(3.14..2)

**Corrollary-3**

for all $i=1\ldots r$; $\mu_i, s, z \in C$; $min\{\mathrm{Re}(a), \mathrm{Re}(s)\} > 0$; $\mathrm{Re}(p_i) > 0$; $\max_{1\le i\le r}|z_i| < 1$

also let $(\gamma_n)_{n\in N_0}$ be a positive sequence such that the following infinite $\sum_{n=0}^{\infty} e^{-\gamma_n t}$

converges for any $t \in R^+$ then

we have the following integral expression

$$\phi_{(\mu_r)}^{(p_r)}(z_1 \ldots z_r; s, a) = \frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1} e^{-(a-\gamma_0+\gamma_n)t}\left[1-e^{-(\gamma_{n+1}-\gamma_n)t}\right]\prod_{i=1}^{r}\left(1-z_i e^{-p_i t}\right)^{-\mu_i} dt$$

…(3.14.3)

(iii) For $r=1$, $\mathrm{Lim}\,\alpha_1 \to 0$, $p_1 = 2$ $\delta_1 = \delta_2 = 1$, $q_1 = 1$, $\rho_1 = 1$, $\gamma_1 = 1$ and, results (3.13.1), (3.13.2), (3.13.3) reduce to the known results due to Garg et.al. ([23], p.313, eq. (2.1); p.314, eqs. (2.4), (2.3)) respectively.

(iv) If we take $r=1$, $p_1 = 2$, $q_1 = 1$, $\gamma_1 = 1$ and $\mathrm{Lim}\,\alpha_1 \to 0$, then results (3.13.1) and (3.13.4) reduce to the known results due to Srivastava and Pogany ([101], p.494, eq.(2.4); p.496, eq. (2.18)) respectively.

(v) If we take $r=1$, $p_1 = p$, $q_1 = q$, $\gamma_1 = 1$ and $\mathrm{Lim}\,\alpha_1 \to 0$, then results (3.13.1) and (3.13.4) reduce to the known results due to Srivastava and Pogany ([101], p.504, eq.(6.4);p.505,eq.(6.9)) respectively.

(vi) For $r=1$, $p_1 = 2$ $q_1 = 1$ and $\gamma_1 = 1$, and $\mathrm{Lim}\,\alpha_1 \to 0$, results (3.13.1), (3.13.2), (3.13.3) and (3.13.4) reduce to the known result due to Gupta ([30],p.169, eq.(3.8.1); pp.171-172, eqs. (3.8.5), (3.8.6), (3.8.4) at $b=0$) respectively.

(vii) For $r=1$, $p_1 = p$, $q_1 = q$ and $\mathrm{Lim}\,\alpha_1 \to 0$, results (3.13.1), (3.13.2), (3.13.3) and (3.13.4) reduce to the known result due to Gupta ([30],pp.172-173, eqs.(3.8.7), (3.8.9), (3.8.10), (3.8.8) at $b=0$) respectively.

(viii) For $\mathrm{Lim}\,\alpha_i \to 0$, results (3.13.1), (3.13.2), (3.13.3) and (3.13.4) reduce to the known result due to Gupta ([30], pp.173-175, eqs. (3.8.11), (3.8.13), (3.8.14), (3.8.12) at $b=0$) respectively.

# CHAPTER-4

# GENERALIZED FRACTIONAL INTEGRAL AND DIFFERENTIAL OPERATORS AND CERTAIN EXTENDED k- GENERALIZED MITTAG-LEFFLER TYPE FUNCTIONS

The main results of this chapter have been published / accepted / communicated for publication as per details below:

1. The generalized fractional calculus of the extended k-generalized Mittag-Leffler function.
   (Communicated for publication)

2. On extended k-generalized Mittag-Leffler type function and its generalized fractional integration and differentiation.
   (Communicated for publication)

## 4.1 INTRODUCTION

This chapter presents a generalized fractional calculus of the extended k-generalized Mittag-Leffler function and extended k-generalized Mittag-Leffler type function via Marichev-Saigo-Maeda integral and differential operators. The Caputo-type Marichev-Saigo-Maeda fractional derivatives are also established for extended k-generalized Mittag-Leffler function and for extended k-generalized Mittag-Leffler type function. The chapter presented into the two parts: Section-A and B.

In section-A, we have obtained the Marichev-Saigo-Maeda integration and differentiation formulae for the extended k-generalized Mittag-Leffler function. The Caputo-type Marichev-Saigo-Maeda fractional derivatives are also established for this extended k-generalized Mittag-Leffler function. In section-B, we consider extended k-generalized Mittag-Leffler type function of arbitrary order. We have obtained Marichev-Saigo-Maeda integration and differentiation and the Caputo-type Marichev-Saigo-Maeda fractional derivatives for this function in this section.

Section-A deals with the six formulae as images of Marichev-Saigo-Maeda operators, two of them deal with the Marichev-Saigo-Maeda fractional integration. While next two formulae deal with the Marichev-Saigo-Maeda fractional differentiation and the rest two formulae are associated with the Caputo-type Marichev-Saigo-Maeda fractional derivatives for the extended k-generalized Mittag-Leffler function. As applications of the main theorems the Riemann-Liouville integral and fractional derivative, Erdelyi-Kober fractional integral and derivative, Saigo and Caputo fractional integral and fractional derivative of extended k-generalized Mittag-Leffler function are shown to be obtained. Certain known and new results for simpler functions are also shown to be derive here.

In the section-B, we have considered the extended k-generalized Mittag-Leffler type function and established six formulae of Marichev-Saigo-Maeda fractional integral and derivative operators and the Caputo-type Marichev-Saigo-Maeda fractional derivatives operators involving this function. As applications of the main theorems the Riemann-Liouville integral and fractional derivative, Erdelyi-Kober fractional integral and derivative, Saigo and Caputo fractional integral and fractional derivative for this extended k-generalized Mittag-Leffler type function are shown to be obtained. Certain known and new results for simpler form of Mittag-Leffler functions are also shown to be derive here in this section.

## 4.2 DEFINITIONS AND PRILIMINARIES

The fractional calculus recently known as calculus of arbitrary order is mainly based upon the following fractional integral and differential operators known as Riemann-Liouville fractional integral and differential operators [48]:

$$\left(I_{0+}^{v} f\right)(x)=\frac{1}{\Gamma(v)}\int_{0}^{x}(x-t)^{v-1} f(t)\,dt \qquad \ldots(4.2.1)$$

$$\left(I_{-}^{v} f\right)(x)=\frac{1}{\Gamma(v)}\int_{x}^{\infty}(t-x)^{v-1} f(t)\,dt \qquad \ldots(4.2.2)$$

$$\left(D_{0+}^{v} f\right)(x)=\left(\frac{d}{dx}\right)^{m}\left(I_{0+}^{m-v} f\right)(x) \qquad \ldots(4.2.3)$$

$$\left(D_{-}^{v} f\right)(x)=\left(-\frac{d}{dx}\right)^{m}\left(I_{-}^{m-v} f\right)(x) \qquad \ldots(4.2.4)$$

where $v\in C, \mathrm{Re}(v)>0, m=\left[\mathrm{Re}(v)\right]+1$ and $x\in R^{+}$.

Marichev [64] and Saigo and Maeda [83] studied the generalized fractional calculus operators with the Appell function $F_3$ in their kernel. These operators are known as Marichev-Saigo-Maeda operators and defined as follows:

Let $v, v', \mu, \mu', \eta \in \mathrm{C}$ with $\mathrm{Re}(\eta)>0$ $x\in R^{+}$ , then the left and right fractional integral operators are defined as follows [83]:

$$\left(I_{0+}^{v,v',\mu,\mu',\eta} f\right)(x)=\frac{x^{-v}}{\Gamma(\eta)}\int_{x}^{\infty}(x-t)^{\eta-1} t^{-v'} F_3\left(v,v',\mu,\mu';\eta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt$$

$$\ldots(4.2.5)$$

$$\left(I_{-}^{v,v',\mu,\mu',\eta} f\right)(x)=\frac{x^{-v'}}{\Gamma(\eta)}\int_{x}^{\infty}(t-x)^{\eta-1} t^{-v} F_3\left(v,v',\mu,\mu';\eta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt$$

$$\ldots(4.2.6)$$

where $F_3$ is defined as follows [96]:

$$F_3(v,v',\mu,\mu';\eta;x,t)=\sum_{m,n=0}^{\infty}\frac{(v)_m (v')_n (\mu)_m (\mu')_n}{(\eta)_{m+n}}\frac{x^m}{m!}\frac{t^n}{n!} \qquad \left(\max\{|x|,|t|\}<1\right)$$

$$\ldots(4.2.7)$$

where $(v)_m$ denotes the Pochhammer symbol defined in term of the Gamma function as below [95]:

$$(v)_m = \frac{\Gamma(v+m)}{\Gamma(v)} = \begin{cases} 1 & ,(m=0) \\ v(v+1)\dots(v+m-1), & m \in N \end{cases} \qquad \dots(4.2.8)$$

The operators defined in (4.2.5) and (4.2.6) are reduced to be the Saigo fractional integral operators defined and represented in the following manner [84]:

$$\left(I_{0+}^{v+\mu,0,-\rho,0,v} f\right)(x) = \left(I_{0+}^{v,\mu,\rho} f\right)(x) = \frac{x^{-v-\mu}}{\Gamma(v)} \int_0^{\infty} (x-t)^{v-1} {}_2F_1\left(v+\mu,-\rho;v;1-\frac{t}{x}\right) f(t)\,dt \qquad \dots(4.2.9)$$

$$\left(I_{-}^{v+\mu,0,-\rho,0,v} f\right)(x) = \left(I_{-}^{v,\mu,\rho} f\right)(x) = \frac{1}{\Gamma(v)} \int_x^{\infty} (t-x)^{v-1} t^{-v-\mu} {}_2F_1\left(v+\mu,-\rho;v;1-\frac{x}{t}\right) f(t)\,dt \qquad \dots(4.2.10)$$

where ${}_2F_1$ is the Gauss hypergeometric series defined as below [96]:

$${}_2F_1(v,\mu;\rho;x) = \sum_{m=0}^{\infty} \frac{(v)_m (\mu)_m}{(\rho)_m} \frac{x^m}{m!}, \qquad (|x|<1) \qquad \dots(4.2.11)$$

For $v, v', \mu, \mu', \eta \in \mathrm{C}$ with $\mathrm{Re}(\eta) > 0$ , $x \in R^+$, the left and right Marichev-Saigo-Maeda fractional differential operators are defined in the following manner [83]:

$$\left(D_{0+}^{v,v',\mu,\mu',\eta} f\right)(x) = \left(I_{0+}^{-v',-v,-\mu',-\mu,-\eta} f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^m \left(I_{0+}^{-v',-v,-\mu'+m,-\mu,-\eta+m} f\right)(x) \qquad \dots(4.2.12)$$

and

$$\left(D_{-}^{v,v',\mu,\mu',\eta} f\right)(x) = \left(I_{-}^{-v',-v,-\mu',-\mu,-\eta} f\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^m \left(I_{-}^{-v',-v,-\mu',-\mu+m,-\eta+m} f\right)(x) \qquad \dots(4.2.13)$$

where $\mathrm{m} = \left[\mathrm{Re}(\eta)\right] + 1$ and $\left[\mathrm{Re}(\eta)\right]$ denotes the integer part of $\mathrm{Re}(\eta)$.

These operators are reduced to be the following Saigo fractional derivative operators [84]:

$$\left(D_{0+}^{v+\mu,0,-\rho,0,v} f\right)(x) = \left(D_{0+}^{v,\mu,\rho} f\right)(x) \quad , \qquad (\rho \in C) \qquad \dots(4.2.14)$$

and

$$\left(D_{-}^{v+\mu,0,-\rho,0,v} f\right)(x) = \left(D_{-}^{v,\mu,\rho} f\right)(x) \quad , \qquad (\rho \in C) \qquad \dots(4.2.15)$$

**Remark-4.2.1**

At $\mu = -\nu$, the operators define in (4.2.9), (4.2.10); (4.2.14) and (4.2.15) respectively reduce to the Riemann-Liouville fractional integral and derivative operators, defined as in (4.2.1),(4.2.2); (4.2.3), (4.2.4) (see also [48]), while at $\mu = 0$ these operators defined in (4.2.9), (4.2.10); (4.2.14) and (4.2.15) reduce to the following Erdelyi-Kober fractional integral and derivative operators [51]:

$$\left(I^{+}_{\rho,\nu} f\right)(x) = \frac{x^{-\nu-\rho}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{\rho} f(t)\,dt \qquad \ldots(4.2.16)$$

$$\left(K^{-}_{\rho,\nu} f\right)(x) = \frac{x^{\rho}}{\Gamma(\nu)} \int_x^{\infty} (t-x)^{\nu-1} t^{-\nu-\rho} f(t)\,dt \qquad \ldots(4.2.17)$$

$$\left(D^{+}_{\rho,\nu} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_{0+}^{-\nu+m,-\nu,\nu+\rho-m} f\right)(x) \qquad \ldots(4.2.18)$$

$$\left(D^{-}_{\rho,\nu} f\right)(x) = \left(-\frac{d}{dx}\right)^m \left(I_{-}^{-\nu+m,-\nu,\nu+\rho} f\right)(x) \qquad \ldots(4.2.19)$$

Where $\rho,\nu \in C,\ \mathrm{Re}(\nu) > 0,\ m = [\mathrm{Re}(\nu)] + 1\ and\ x \in R^{+}$.

The following fractional integral and derivative formulae obtained in view of (4.2.5), (4.2.6); (4.2.12), (4.2.13) are required here [83]:

$$\left(I_{0+}^{\nu,\nu',\mu,\mu',\eta} t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda+\eta-\nu-\nu'-\mu)\Gamma(\lambda+\mu'-\nu')}{\Gamma(\lambda+\mu')\Gamma(\lambda+\eta-\nu-\nu')\Gamma(\lambda+\eta-\nu'-\mu)} x^{\lambda-\nu-\nu'+\eta-1}$$

$$\ldots(4.2.20)$$

where $\nu,\nu',\mu,\mu',\eta \in C$ and

$\mathrm{Re}(\eta) > 0, \mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(\nu+\nu'+\mu-\eta), \mathrm{Re}(\nu'+\mu')\}$;

$$\left(I_{-}^{\nu,\nu',\mu,\mu',\eta} t^{\lambda-1}\right)(x) = \frac{\Gamma(1-\lambda-\mu)\Gamma(1-\lambda-\eta+\nu+\nu')\Gamma(1-\lambda+\nu+\mu'-\eta)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\nu-\mu)\Gamma(1-\lambda+\nu+\nu'+\mu'-\eta)} x^{\lambda-\nu-\nu'+\eta-1}$$

$$\ldots(4.2.21)$$

where $\nu,\nu',\mu,\mu',\eta \in C$, $\mathrm{Re}(\eta) > 0$ and

$\mathrm{Re}(\lambda) < 1 + \max\{\mathrm{Re}(-\mu), \mathrm{Re}(\nu+\nu'-\eta), \mathrm{Re}(\nu+\mu'-\eta)\}$.

$$\left(D_{0+}^{\nu,\nu',\mu,\mu',\eta} t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda-\eta+\nu+\nu'+\mu')\Gamma(\lambda-\mu+\nu)}{\Gamma(\lambda-\mu)\Gamma(\lambda-\eta+\nu+\nu')\Gamma(\lambda-\eta+\nu+\mu')} x^{\lambda+\nu+\nu'-\eta-1}$$

$$\ldots(4.2.22)$$

where $v, v', \mu, \mu', \eta \in C$ and

$\operatorname{Re}(\eta) > 0, \operatorname{Re}(\lambda) > \max\{0, \operatorname{Re}(\eta - v - v' - \mu'), \operatorname{Re}(\mu - v)\}$;

$$\left(D_{-}^{v,v',\mu,\mu',\eta} t^{\lambda-1}\right)(x) = \frac{\Gamma(1-\lambda+\mu')\Gamma(1-\lambda-v'-\mu+\eta)\Gamma(1-\lambda-v-v'+\eta)}{\Gamma(1-\lambda)\Gamma(1-\lambda+v'+\mu')\Gamma(1-\lambda-v-v'-\mu+\eta)} x^{\lambda+v+v'-\eta-1} \quad \ldots(4.2.23)$$

where $v, v', \mu, \mu', \eta \in C$ and

$\operatorname{Re}(\eta) > 0, \operatorname{Re}(\lambda) < 1 + \max\{\operatorname{Re}(\mu'), \operatorname{Re}(\eta - v - v'), \operatorname{Re}(\eta - v' - \mu)\}$.

Recently Rao et.al [82] introduced and studied the Caputo-type fractional derivative that have the Gauss hypergeometric function in the kernel these fractional derivatives are defined in the following manner:

Let $\mu, v, \alpha \in C$ with $\operatorname{Re}(v) > 0, x \in R^{+}$ then the left and right Caputo-type fractional differential operators are defined as follows.

$$\left({}^{C}D_{0+}^{\mu,v,\alpha} f\right)(x) = \left(I_{0+}^{-v+[\operatorname{Re}(v)]+1, -\mu-[\operatorname{Re}(v)]-1, v+\alpha-[\operatorname{Re}(v)]-1} f^{([\operatorname{Re}(v)]+1)}\right)(x) \quad \ldots(4.2.24)$$

and

$$\left({}^{C}D_{-}^{\mu,v,\alpha} f\right)(x) = (-1)^{([\operatorname{Re}(v)]+1)} \left(I_{0+}^{-v+[\operatorname{Re}(v)]+1, -\mu-[\operatorname{Re}(v)]-1, v+\alpha} f^{[\operatorname{Re}(v)]+1}\right)(x) \quad \ldots(4.2.25)$$

The Caputo-type Marichev-Saigo-Maeda fractional differential operators are recently studied by Kataria and Vellaisamy [46]:

Let $v, v', \mu, \mu', \xi \in C$ with $\operatorname{Re}(\xi) > 0, x \in R^{+}$ then the left and right Caputo-type Marichev-Saigo-Maeda fractional differential operators are defined as follows,

$$\left({}^{C}D_{0+}^{v,v',\mu,\mu',\xi} f\right)(x) = \left(I_{0+}^{-v',-v,-\mu'+[\operatorname{Re}(v)]+1, -\mu-\xi+[\operatorname{Re}(v)]+1} f^{([\operatorname{Re}(v)]+1)}\right)(x) \quad \ldots(4.2.26)$$

and

$$\left({}^{C}D_{-}^{v,v',\mu,\mu',\xi} f\right)(x) = (-1)^{[\operatorname{Re}(v)+1]} \left(I_{-}^{-v',-v,-\mu',-\mu+\operatorname{Re}(v)+1, -\xi+[\operatorname{Re}(v)]-1, v+\alpha} f^{[\operatorname{Re}(v)]+1}\right)(x) \quad \ldots(4.2.27)$$

The following known results for the Caputo-type Marichev-Saigo-Maeda fractional differentiation are required here [46]:

$$\left({}^{C}D_{0+}^{v,v',\mu,\mu',\xi} t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda-\xi+v+v'+\mu'-b)\Gamma(\lambda-\mu+v-b)}{\Gamma(\lambda-\mu-b)\Gamma(\lambda-\xi+v+v')\Gamma(\lambda-\xi+v+\mu'-b)} x^{\lambda+v+v'-\xi-1} \quad \ldots(4.2.28)$$

where $v,v',\mu,\mu',\xi\in\mathrm{C}$, $\mathrm{Re}(\lambda)-b>\max\{0,\mathrm{Re}(\xi-v-v'-\mu'),\mathrm{Re}(\mu-v)\}$, and $b=[\mathrm{Re}(\xi)]+1$

$$\left({}^{C}D_{-}^{v,v',\mu,\mu',\xi}t^{-\lambda}\right)(x)=\frac{\Gamma(\lambda+\mu'+b)\Gamma(\lambda-v'-\mu+\xi+b)\Gamma(\lambda-v-v'+\xi)}{\Gamma(\lambda)\Gamma(\lambda-v'+\mu'+b)\Gamma(\lambda-v-v'-\mu+\xi+b)}x^{v+v'-\xi-\lambda} \quad \ldots(4.2.29)$$

where $v,v',\mu,\mu',\xi\in\mathrm{C}$, $\mathrm{Re}(\lambda)+b>\max\{\mathrm{Re}(-\mu),\mathrm{Re}(v'+\mu-\xi),\mathrm{Re}(v+v'-\xi)+b\}$ and $b=[\mathrm{Re}(\xi)]+1$.

The elementary Mittag-Leffler function is defined and represented as follows [66],

$$E_l(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(nl+1)}\quad,\qquad (x\in C,l>0) \quad \ldots(4.2.30)$$

A generalization of this function was given by Wiman [113],

$$E_{l,m}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(nl+m)} \quad \ldots(4.2.31)$$

where $x,l,m\in C;\mathrm{Re}(l)>0,\mathrm{Re}(m)>0$

The above definition of Mittag-Leffler function is extended by Prabhakar [78]:

$$E_{l,m}^{\rho}(x)=\sum_{n=0}^{\infty}\frac{(\rho)_n}{\Gamma(nl+m)}\frac{x^n}{n!} \quad \ldots(4.2.32)$$

where $x,l,m,\rho\in C;\mathrm{Re}(l)>0,\mathrm{Re}(m)>0$

Recently Shukla and Prajapati [93] studied the following four parameter Mittag-Leffler function:

$$E_{l,m}^{\rho,\sigma}(x)=\sum_{n=0}^{\infty}\frac{(\rho)_{n\sigma}}{\Gamma(nl+m)}\frac{x^n}{n!} \quad \ldots(4.2.33)$$

where $x,l,m,\rho\in C;\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0$ and $\sigma\in(0,1)\cup N$.

The Mittag-Leffler function is further extended using the k-Gamma function [13]. It is defined and represented as follows by G.A. Dorrego et al. [17]:

$$E_{k,l,m}^{\rho}(x)=\sum_{n=0}^{\infty}\frac{(\rho)_{n,k}}{\Gamma_k(nl+m)}\frac{x^n}{n!} \quad \ldots(4.2.34)$$

$k>0$, $x,l,m,\rho\in C;\operatorname{Re}(l)>0,\operatorname{Re}(m)>0,\operatorname{Re}(\rho)>0$,

where $(\rho)_{n,k}=\dfrac{\Gamma_k(\rho+n)}{\Gamma_k(\rho)}$ …(4.2.35)

and $\Gamma_k(x)=\int_0^1 e^{-t}t^{\frac{x}{k}-1}dt,\qquad k>0$. …(4.2.36)

For the more the generalized Mittag-Leffler function and their applications, we refer to [2,26,35,42].

Recently Rahman et al. [79] studied the following extension of k-Mittag-Leffler function:

$$E_{k,l,m}^{\rho;\mathrm{c}}(x;p)=\sum_{n=0}^{\infty}\frac{B_k(\rho+nk,\mathrm{c}-\rho;\mathrm{p})(c)_{n,k}}{B_k(\rho,\mathrm{c}-\rho)\Gamma_k(nl+m)}\frac{x^n}{n!} \qquad \text{…(4.2.37)}$$

For $\dfrac{(\rho)_{n,k}}{(c)_{n,k}}=\dfrac{B_k(\rho+nk,\mathrm{c}-\rho)}{B_k(\rho,\mathrm{c}-\rho)}$ …(4.2.38)

where $B_k(x,y)$ is the k-Beta function defined in [13]:

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \qquad \text{…(4.2.39)}$$

and $B_k(l,m;p)$ is an extension of the k-Beta function defined as follows [71] (see also [70]):

$$B_k(l,m;p)=\frac{1}{k}\int_0^1 t^{\frac{l}{k}-1}(1-t)^{\frac{m}{k}-1}e^{-\frac{p^k}{kt(1-t)}}dt \qquad \text{…(4.2.40)}$$

The above extended function is further studied by Jain et al. [45]. This extended k-generalized Mittag-Leffler function is defined and represented as follows [45, p.473, eq. (10)]:

$$E_{k,l,m}^{\rho,\sigma;\mathrm{c}}(x;p)=\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{B_k(\rho,\mathrm{c}-\rho)}\frac{(c)_{n\sigma,k}}{\Gamma_k(nl+m)}\frac{x^n}{n!} \qquad \text{…(4.2.41)}$$

where $k>0;x,l,m,\rho\in C;\operatorname{Re}(l)>0,\operatorname{Re}(m)>0,\operatorname{Re}(\rho)>0,\sigma\in(0,1)\cup N;\ p\geq 0.$

## SECTION-A

**4.3** Recently Bin Saad and Younis [5] studied Marichev-Saigo-Maeda fractional integration and differentiation and the Caputo-type Marichev-Saigo-Maeda fractional derivative of the extended Mittag-Leffler- type function of arbitrary order. Motivated by the above cited work due to Bin Saad and Younis, we established six fractional integral and differentiation formulae for the extended k-generalized Mittag-Leffler function defined in (4.2.41) in this section. First two formulae deal with the Marichev-Saigo-Maeda fractional integral operators $\left(I_{0+}^{v,v',\mu,\mu',\eta} f\right)(x)$ and $\left(I_{-}^{v,v',\mu,\mu',\eta} f\right)(x)$, while the next two formulae are associated to the Marichev-Saigo-Maeda fractional derivative operators $\left(D_{0+}^{v,v',\mu,\mu',\eta} f\right)(x)$ and $\left(D_{-}^{v,v',\mu,\mu',\eta} f\right)(x)$ of this k-generalized Mittag-Leffler function. The remaining two formulae associated to the Caputo-type Marichev-Saigo-Maeda fractional derivatives operators $\left({}^{C}D_{0+}^{v,v',\mu,\mu',\xi} f\right)(x)$ and $\left({}^{C}D_{-}^{v,v',\mu,\mu',\xi} f\right)(x)$ are also established for the above extended k-generalized Mittag-Leffler function (4.2.41). As applications of the main theorems established in this section, Riemann-Liouville, Erdelyi-Kober fractional integral and derivative, Saigo and Caputo fractional integral and fractional derivative of extended k-generalized Mittag-Leffler function are shown to be obtained and certain known and new results for simpler functions are also shown to be derived here.

## 4.4 MAIN THEOREMS

**Marichev-Saigo-Maeda Fractional Integrals of the Function** $E_{k,l,m}^{\rho,\sigma;c}(t;p)$

Two fractional integrals of the extended k-generalized Mittag-Leffler function are established here by applying left and right sided Marichev-Saigo-Maeda fractional integral operators defined in (4.2.5) and (4.2.6). These image formulae are given in Theorem-1 and Theorem-2 as below:

**Theorem-1**

For $v, v', \mu, \mu', \eta, \lambda, t, l, m, \mathrm{c} \in \mathrm{C}$ with

$\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(v+v'+\mu-\eta), \mathrm{Re}(v'-\mu')\} > 0$,

$\mathrm{Re}(\eta) > 0, \mathrm{Re}(c) > 0, k > 0, \mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0$ , $\sigma \in (0,1) \cup N; \mathrm{p} \in R_0^+$,

we have the following Marichev-Saigo-Maeda left fractional integral,

$$\left(I_{0+}^{v,v',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(t;p\right)\right]\right)(x)=x^{-v-v'+\eta+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}$$

$$\frac{\Gamma(\lambda+n)\Gamma(-v'+\mu'+\lambda+n)\Gamma(-v-v'-\mu+\eta+\lambda+n)}{\Gamma(\lambda+\mu'+\mathrm{n})\Gamma(-v-v'+\eta+\lambda+n)\Gamma(-v'-\mu+\eta+\lambda+n)}\frac{x^n}{n!}\quad \ldots(4.4.1)$$

**Theorem-2**

For $v,v',\mu,\mu',\eta,\lambda,\rho,l,m,\mathrm{c}\in\mathrm{C}$ with

$\mathrm{Re}(\lambda)<1+\min\left\{\mathrm{Re}(-\mu),\mathrm{Re}(v+v'-\eta),\mathrm{Re}(v+\mu'-\eta)\right\}$,

$\mathrm{Re}(\eta)>0,\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0$ and $k>0,\mathrm{p}\in R_0^+$

we have the following Marichev-Saigo-Maeda right fractional integral,

$$\left(I_{-}^{v,v',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(\frac{1}{t};p\right)\right]\right)(x)=x^{-v-v'+\eta+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}$$

$$\frac{\Gamma(-\mu-\lambda+n+1)\Gamma(v+v'-\eta-\lambda+n+1)\Gamma(v+\mu'-\eta-\lambda+n+1)}{\Gamma(-\lambda+\mathrm{n}+1)\Gamma(v-\mu-\lambda+n+1)\Gamma(v+v'+\mu'-\eta-\lambda+n+1)}\left(\frac{1}{x}\right)^n\frac{1}{n!}$$

$$\ldots(4.4.2)$$

**Marichev-Saigo-Maeda Fractional Derivative Of The Function $E_{k,l,m}^{\rho,\sigma;c}\left(t;p\right)$**

The two fractional derivative image formulae for the extended k-generalized Mittag-Leffler function are obtained here by applying the left and right Marichev-Saigo-Maeda fractional derivative operators in view of (4.2.12) and (4.2.13). These formulae are given in the following Theorem-3 and Theorem-4 as below:

**Theorem-3**

For $v,v',\mu,\mu',\xi,\lambda,l,m,\rho,\mathrm{c}\in\mathrm{C}$ with $\mathrm{Re}(\lambda)>\max\left\{0,\mathrm{Re}(-v-v'-\mu'+\xi),\mathrm{Re}(\mu-v)\right\}$

, $\mathrm{Re}(\xi)>0,\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0$ and $k>0$, $\mathrm{p}\in R_0^+$,

$\sigma\in(0,1)\cup N$,

we have the Marichev-Saigo-Maeda left fractional derivative,

$$\left(D_{0+}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(\mathrm{t};p\right)\right]\right)(x)=x^{\lambda+v+v'-\xi-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}\frac{x^n}{n!}$$

$$\frac{\Gamma(\lambda+n)\Gamma(v-\mu+\lambda+n)\Gamma(v+v'+\mu'-\xi+\lambda+n)}{\Gamma(-\mu+\lambda+\mathrm{n})\Gamma(v+v'-\xi+\lambda+n)\Gamma(v+\mu'-\xi+\lambda+n)}\quad \ldots(4.4.3)$$

**Theorem-4**

For $v,v',\mu,\mu',\xi,\lambda,l,m,\rho,c\in\mathbb{C}$, $\sigma\in(0,1)\cup N$

$\mathrm{Re}(\lambda)<1+\min\{\mathrm{Re}(\mu'),\mathrm{Re}(-v-v'+\xi),\mathrm{Re}(-v'-\mu+\xi)\}$,

$\mathrm{Re}(\xi)>0,\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0$ and $k>0, \mathrm{p}\in R_0^+$

we have the Marichev-Saigo-Maeda right fractional derivative,

$$\left(D_{-}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(\frac{1}{t};p\right)\right]\right)(x)=x^{v+v'-\xi+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}\left(\frac{1}{x}\right)^n\frac{1}{n!}$$

$$\frac{\Gamma(\mu'-\lambda+n+1)\Gamma(-v-v'+\xi-\lambda+n+1)\Gamma(-v'-\mu+\xi-\lambda+n+1)}{\Gamma(-\lambda+n+1)\Gamma(-v'+\mu'-\lambda+n+1)\Gamma(-v-v'-\mu+\xi-\lambda+n+1)} \quad \ldots(4.4.4)$$

**Caputo-Type Marichev-Saigo-Maeda Fractional Derivative of the Function $E_{k,l,m}^{\rho,\sigma;c}(t;p)$**

In the next two Theorem-5 and Theorem-6, two fractional derivatives of the extended k-generalized Mittag-Leffler function are established by applying left and right sided Caputo-type Marichev-Saigo-Maeda fractional derivatives operators defined in (4.2.26) and (4.2.27). The Theorem-5 and Theorem-6 are given as below:

**Theorem-5**

For $v,v',\mu,\mu',\xi,\lambda,l,m,\rho,c\in\mathbb{C}$, $\sigma\in(0,1)\cup N$,

$\mathrm{Re}(\lambda)-b>\max\{0,\mathrm{Re}(-v-v'-\mu'+\xi),\mathrm{Re}(\mu-v)\}$, $b=[\mathrm{Re}(\xi)]+1$,

$\mathrm{Re}(\xi)>0,\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0$ and $k>0$, $\mathrm{p}\in R_0^+$,

then we have the left Caputo-type Marichev-Saigo-Maeda fractional derivative as follows.

$$\left({}^{C}D_{O+}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}(\mathrm{t};p)\right]\right)(x)=x^{v+v'-\xi+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}\frac{x^n}{n!}$$

$$\frac{\Gamma(\lambda+n)\Gamma(v-\mu+\lambda-b+n)\Gamma(v+v'+\mu'-\xi+\lambda-b+n)}{\Gamma(-\mu+\lambda-b+\mathrm{n})\Gamma(v+v'-\xi+\lambda+n)\Gamma(v+\mu'-\xi+\lambda-b+n)} \quad \ldots(4.4.5)$$

**Theorem-6**

For $v, v', \mu, \mu', \xi, \lambda, l, m, \rho, \mathrm{c} \in \mathrm{C}$, $\sigma \in (0,1) \cup N$,

$\mathrm{Re}(\lambda) + b > \max\{-\mathrm{Re}(\mu'), \mathrm{Re}(v' + \mu - \xi), \mathrm{Re}(v, v' - \xi) + b\}$,

$\mathrm{b} = [\mathrm{Re}(\xi)] + 1, \mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$ and $k > 0, \mathrm{p} \in R_0^+$

then we have the right Caputo-type Marichev-Saigo-Maeda fractional derivative as follows:

$$\left({}^{C}D_{-}^{v,v',\mu,\mu',\xi}\left[t^{-\lambda}E_{k,l,m}^{\rho,\sigma;\mathrm{c}}\left(\frac{1}{t};p\right)\right]\right)(x) = x^{v+v'-\xi-\lambda}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}\left(\frac{1}{x}\right)^n\frac{1}{n!}$$

$$\frac{\Gamma(\mu'+\lambda+b+n)\Gamma(-v-v'+\xi+\lambda+n)\Gamma(-v'-\mu+\xi+\lambda+b+n)}{\Gamma(\lambda+n)\Gamma(-v'+\mu'+\lambda+b+n)\Gamma(-v-v'-\mu+\xi-\lambda+b+n)} \quad \ldots(4.4.6)$$

**OUTLINE OF PROOFS**

**Proof of Thorem-1**

To prove the Theorem-1, let the left hand side of (4.4.1) is denoted by $\Delta_1$, i.e.

$$\Delta_1 = \left(I_{0+}^{v,v',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}(t;p)\right]\right)(x)$$

Now on using the definition given in (4.2.41) and then on arranging the order of summation and integration we have

$$\Delta_1 = \sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})(c)_{n\sigma,k}}{B_k(\rho, c-\rho)\Gamma_k(nl+m)\; n!}\left(I_{0+}^{v,v',\mu,\mu',\eta}t^{\lambda+n-1}\right)(x)$$

Now on applying the formulae (4.2.20), we get

$$\Delta_1 = \sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})(c)_{n\sigma,k}}{B_k(\rho, c-\rho)\Gamma_k(nl+m)\; n!}$$

$$\frac{\Gamma(\lambda+n)\Gamma(-v'+\mu'+\lambda+n)\Gamma(-v-v'-\mu+\eta+\lambda+n)}{\Gamma(\lambda+\mu'+\mathrm{n})\Gamma(-v-v'+\eta+\lambda+n)\Gamma(-v'-\mu+\eta+\lambda+n)}x^{-v-v'+\eta+\lambda+n-1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result in (4.4.1). This completes the proof of Theorem-1.

**Proof of Theorem-2**

To prove the Theorem-2, let the left hand side of (4.4.2) is denoted by $\Delta_2$, i.e.

$$\Delta_2 = \left( I_{-}^{v,v',\mu,\mu',\eta} \left[ t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;c} \left( \frac{1}{t}; p \right) \right] \right)(x)$$

Now on using the definition given in (4.2.41) and then on arranging the order of summation and integration we have

$$\Delta_2 = \sum_{n=0}^{\infty} \frac{B_k(\rho + n\sigma k, c-\rho; p)(c)_{n\sigma,k}}{B_k(\rho, c-\rho)\Gamma_k(nl+m)n!} \left( I_{-}^{v,v',\mu,\mu',\eta} t^{\lambda-n-1} \right)(x)$$

Now on applying the formulae (4.2.21), we get

$$\Delta_2 = \sum_{n=0}^{\infty} \frac{B_k(\rho + n\sigma k, c-\rho; p)(c)_{n\sigma,k}}{B_k(\rho, c-\rho)\Gamma_k(nl+m)n!} \frac{\Gamma(-\mu-\lambda+n+1)}{\Gamma(-\lambda+n+1)}$$

$$\frac{\Gamma(v+v'-\eta-\lambda+n+1)\Gamma(v+\mu'-\eta-\lambda+n+1)}{\Gamma(v-\mu-\lambda+n+1)\Gamma(v+v'+\mu'-\eta-\lambda+n+1)} x^{\lambda-n-v-v'+\eta-1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result in (4.4.2). This completes the proof of Theorem-2.

**Proof of Theorem-3**

To prove the Theorem-3, let the left hand side of (4.4.3) is denoted by $\Delta_3$, then

$$\Delta_3 = \left( D_{0+}^{v,v',\mu,\mu',\xi} \left[ t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;c}(t; p) \right] \right)(x)$$

Now on using the definition (4.2.41) and applying the left Marichev-Saigo-Maeda fractional derivative and on changing the order of differentiation and summation, we have

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{B_k(\rho + n\sigma k, c-\rho; p)(c)_{n\sigma,k}}{B_k(\rho, c-\rho)\Gamma_k(nl+m)n!} \left( D_{0+}^{v,v',\mu,\mu',\xi} t^{\lambda+n-1} \right)(x)$$

On applying the formulae (4.2.22), it takes the following form

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{B_k(\rho + n\sigma k, c-\rho; p)(c)_{n\sigma,k}}{B_k(\rho, c-\rho)\Gamma_k(nl+m)n!}$$

$$\frac{\Gamma(\lambda+n)\Gamma(v+v'+\mu'-\xi+\lambda+n)\Gamma(v-\mu+\lambda+n)}{\Gamma(-\mu+\lambda+n)\Gamma(v+v'-\xi+\lambda+n)\Gamma(v+\mu'-\xi+\lambda+n)} x^{v+v'-\xi+\lambda+n-1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result in (4.4.3). This completes the proof of Theorem-3.

**Proof of Theorem-4**

To prove the Theorem-4, let the left hand side of (4.4.4) is denoted by $\Delta_4$ i.e.,

$$\Delta_4 = \left( D_{-}^{v,v',\mu,\mu',\xi} \left[ t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;c} \left( \frac{1}{t}; p \right) \right] \right)(x)$$

Now on using the definition (4.2.41) and applying the right Marichev-Saigo-Maeda fractional derivative and on changing the order of differentiation and summation, we get

$$\Delta_4 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, c-\rho; p\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(nl+m\right)} \left( D_{-}^{v,v',\mu,\mu',\xi} t^{\lambda-n-1} \right)(x)$$

On applying the formulae (4.2.23), it takes the following form

$$\Delta_4 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, c-\rho; p\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(nl+m\right)n!} \frac{\Gamma\left(\mu'-\lambda+n+1\right)}{\Gamma\left(-\lambda+n+1\right)}$$

$$\frac{\Gamma\left(-v-v'+\xi-\lambda+n+1\right)\Gamma\left(-v'-\mu+\xi-\lambda+n+1\right)}{\Gamma\left(-v'+\mu'-\lambda+n+1\right)\Gamma\left(-v-v'-\mu+\xi-\lambda+n+1\right)} x^{\lambda-n+v+v'-\xi-1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result in (4.4.4). This completes the proof of Theorem-4

**Proof of Theorem-5**

To prove the Theorem-4, let the left hand side of (4.4.5) is denoted by $\Delta_5$ i.e.

$$\Delta_5 = \left( {}^{C}D_{0+}^{v,v',\mu,\mu',\xi} \left[ t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;c} \left( t; p \right) \right] \right)(x)$$

Now using the definition (4.2.41) and on changing the order of differentiation and summation, we have

$$\Delta_5 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, c-\rho; p\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(nl+m\right)} \left( {}^{C}D_{0+}^{v,v',\mu,\mu',\xi} t^{\lambda+n-1} \right)(x)$$

On using the formulae (4.2.28), it can be written as

$$\Delta_5 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, c-\rho; p\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(nl+m\right)}$$

$$\frac{\Gamma\left(\lambda+n\right)\Gamma\left(v-\mu+\lambda-b+n\right)\Gamma\left(v+v'+\mu'-\xi+\lambda-b+n\right)}{\Gamma\left(-\mu+\lambda-b+n\right)\Gamma\left(v+v'-\xi+\lambda+n\right)\Gamma\left(v+\mu'-\xi+\lambda-b+n\right)} x^{\lambda+n+v+v'-\xi-1}$$

Now in view of (4.2.35) and (4.2.39), we at once get the required result in (4.4.5). This completes the proof of Theorem-5

**Proof of Theorem-6**

To prove the Theorem-6, let the left hand side of (4.4.6) is denoted by $\Delta_6$ i.e.,

$$\Delta_6 = \left( {}^{C}D_{-}^{v,v',\mu,\mu',\xi} \left[ t^{-\lambda} E_{k,l,m}^{\rho,\sigma;c} \left( \frac{1}{t}; p \right) \right] \right)(x)$$

Now using the definition (4.2.41) and on changing the order of differentiation and summation, we get

$$\Delta_6 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho; \mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(nl+m\right)} \left( {}^{C}D_{-}^{v,v',\mu,\mu',\xi} t^{-(\lambda+n)} \right)(x)$$

On using the formulae (4.2.29) it can be written as

$$\Delta_6 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho; \mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(nl+m\right)\ n!}$$

$$\frac{\Gamma\left(\mu'+\lambda+b+n\right)\Gamma\left(-v-v'+\xi+\lambda+n\right)\Gamma\left(-v'-\mu+\xi+\lambda+b+n\right)}{\Gamma\left(\lambda+n\right)\Gamma\left(-v'+\mu'+\lambda+b+n\right)\Gamma\left(-v-v'-\mu+\xi-\lambda+b+n\right)} x^{v+v'-\xi-\lambda-n}$$

Now in view of (4.2.35) and (4.2.39), we at once get the required result (4.4.6). This completes the proof of Theorem-6.

## 4.5 APPLICATIONS

As application of the main results, the corresponding formulae for Saigo operators, Erdelyi-Kober operators and Riemann-Liouville operators are obtained and given as following corollaries.

(i) Application of the results (4.4.1) and (4.2.2); we obtain the following fractional integral formulae for Saigo fractional integral operator.

**Corollary-1**

For $v, \mu, t, l, m, \mathrm{c}, \rho, \lambda, \alpha \in \mathbb{C}$, $\mathrm{Re}(v) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0$ ,

$\sigma \in (0,1) \cup N; k > 0, \mathrm{p} \in R_0^+$ with $\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(\mu-\alpha)\}$,

we have the following left fractional integral formula.

$$\left(I_{0+}^{\nu,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(t;p\right)\right]\right)(x)=x^{-\mu+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,c-\rho;p\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}$$

$$\frac{\Gamma(\lambda+n)\Gamma(-\mu+\alpha+\lambda+n)}{\Gamma(-\mu+\lambda+n)\Gamma(\nu+\alpha+\lambda+n)}\frac{x^n}{n!} \qquad \ldots(4.5.1)$$

**Proof of (4.5.1)**

The result (4.5.1) is easily established following similar lines as in the proof of result in (4.4.1) in view of (4.2.9).

**Corollary-2**

For $\nu,\mu,\alpha,\lambda,l,m,\rho,c\in\mathbb{C},$

$\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0,\mathrm{Re}(\nu)>0,\ \sigma\in(0,1)\cup N$

$\mathrm{Re}(\lambda)<1+\min\{\mathrm{Re}(\mu),\mathrm{Re}(\alpha)\}\quad k>0,\mathrm{p}\in R_0^+,$

we have the following right fractional integral formula

$$\left(I_{-}^{\nu,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(\frac{1}{t};p\right)\right]\right)(x)=x^{-\mu+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,c-\rho;p\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}$$

$$\frac{\Gamma(\alpha-\lambda+n+1)\Gamma(\mu-\lambda+n+1)}{\Gamma(-\lambda+n+1)\Gamma(\nu+\mu+\alpha-\lambda+n+1)}\left(\frac{1}{x}\right)^n\frac{1}{n!}$$

…(4.5.2)

**Proof of (4.5.2)**

The result (4.5.2) is easily established following similar lines as in the proof of (4.4.2) in view of (4.2.10).

The results (4.5.1) and (4.5.2) in turn at $\mu=0$ provide the following fractional integrals related to Erdelyi-Kober operators defined in (4.2.16) and (4.2.17):

**Corollary-3**

For $\nu,\mu,t,l,m,c,\rho,\lambda,\alpha\in\mathbb{C}$, $\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,$

$\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0,\mathrm{Re}(\nu)>0$, $\sigma\in(0,1)\cup N;\mathrm{p}\in R_0^+$ and $k>0$

$\mathrm{Re}(\lambda)>-\mathrm{Re}(\alpha),$

we have the following left fractional integral formula.

$$\left(I_{\alpha,v}^{+}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}(t;p)\right]\right)(x)=x^{\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}$$

$$\frac{\Gamma(\alpha+\lambda+n)}{\Gamma(v+\alpha+\lambda+n)}\frac{x^n}{n!} \quad \ldots(4.5.3)$$

**Corollary-4**

For $v,\alpha,\lambda,l,m,\rho,c\in C$, $\operatorname{Re}(l)>0,\operatorname{Re}(m)>0,\operatorname{Re}(\rho)>0,\operatorname{Re}(v)>0,$

$\sigma\in(0,1)\cup N$ $\operatorname{Re}(\lambda)<1+\operatorname{Re}(\alpha)$ $k>0, p\in R_0^{+}$

we have the following right fractional integral formula.

$$\left(K_{\alpha,v}^{-}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(\frac{1}{t};p\right)\right]\right)(x)$$

$$=x^{\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}\frac{\Gamma(\alpha-\lambda+n+1)}{\Gamma(\alpha+v-\lambda+n+1)}\left(\frac{1}{x}\right)^n\frac{1}{n!}$$

$$\ldots(4.5.4)$$

At $\mu=-v$ the result (4.5.1) and (4.5.2) reduce to the Riemann-Liouville fractional integral of extended k-generalized Mittag-Leffler function. These are given as below in Corollary-5 and Corollary-6.

**Corollary-5**

For $t,v,\lambda,c,l,m,\rho\in C$, $\operatorname{Re}(l)>0,\operatorname{Re}(m)>0,$

$\operatorname{Re}(\rho)>0,\operatorname{Re}(c)>0,\operatorname{Re}(v)>0$, $\sigma\in(0,1)\cup N$; $p\in R_0^{+}$ and $k>0$ $\operatorname{Re}(\lambda)>0$

we have the following left fractional integral formula.

$$\left(I_{0+}^{v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}(t;p)\right]\right)(x)$$

$$=x^{v+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}\frac{\Gamma(\lambda+n)}{\Gamma(v+\lambda+n)}\frac{x^n}{n!} \quad \ldots(4.5.5)$$

**Corollary-6**

For $v,\lambda,l,m,\rho,c\in C$,

$\operatorname{Re}(l)>0,\operatorname{Re}(m)>0,\operatorname{Re}(\rho)>0,\operatorname{Re}(c)>0,\operatorname{Re}(v)>0,$ $\operatorname{Re}(v)<1-\operatorname{Re}(\lambda)$

$k>0, p\in R_0^{+}$, $\sigma\in(0,1)\cup N$, we have the following right fractional integral formula.

$$\left(I_{-}^{v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c}}\left(\frac{1}{t};p\right)\right]\right)(x)$$

$$=x^{v+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}\frac{\Gamma\left(-v-\lambda+n+1\right)}{\Gamma\left(-\lambda+n+1\right)}\left(\frac{1}{x}\right)^n\frac{1}{n!}$$

…(4.5.6)

(ii) Application of results (4.4.3) and (4.4.4): we obtain the following fractional derivative formulae for Saigo differential operator:

**Corollary-7**

For $v,\mu,\alpha,\lambda,l,m,\rho,\mathrm{c}\in\mathbb{C}$, $\sigma\in(0,1)\cup N$,

$\mathrm{Re}(\lambda)>-\min\{0,\mathrm{Re}(v+\mu+\alpha)\}$, $\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0$

and $k>0$, $\mathrm{p}\in\boldsymbol{R}_0^{+}$,

we have the following left fractional formula.

$$\left(D_{0+}^{v,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(\mathrm{t};p\right)\right]\right)(x)=x^{\mu+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}\frac{x^n}{n!}$$

$$\frac{\Gamma\left(\lambda+n\right)\Gamma\left(v+\mu+\alpha+\lambda+n\right)}{\Gamma\left(\alpha+\lambda+\mathrm{n}\right)\Gamma\left(\mu+\lambda+n\right)} \qquad \ldots(4.5.7)$$

**Proof of (4.5.7)**

The result (4.5.7) is easily established following similar lines as in the proof of result (4.4.3) in view of (4.2.14).

**Corollary-8**

For $v,\mu,\alpha,\lambda,l,m,\rho,\mathrm{c}\in\mathbb{C}$, $\sigma\in(0,1)\cup N$,

$\mathrm{Re}(\lambda)<1+\min\{\mathrm{Re}(-\mu)-b,\mathrm{Re}(v+\alpha)\}$, $b=\left[\mathrm{Re}(v)\right]+1$,

$\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0$ and $k>0$, $\mathrm{p}\in\boldsymbol{R}_0^{+}$,

we have the following right fractional formula.

$$\left(D_{-}^{v,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}\left(\mathrm{t};p\right)\right]\right)(x)=x^{\mu+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)n!}\left(\frac{1}{x}\right)^n$$

$$\frac{\Gamma\left(-\mu-\lambda+1+n\right)\Gamma\left(v+\alpha-\lambda+1+n\right)}{\Gamma\left(-\lambda+1+\mathrm{n}\right)\Gamma\left(-\mu+\alpha-\lambda+1+n\right)} \qquad \ldots(4.5.8)$$

**Proof of (4.5.8)**

The result (4.5.8) is easily established following similar lines as in the proof of result (4.4.4) in view of (4.2.15).

The results (4.5.7) and (4.5.8) in turn at $\mu = 0$ provides the following fractional derivatives related to Erdelyi-Kober operator defined in (4.2.18) and (4.2.19):

**Corollary-9**

For $\alpha, v, \lambda, l, m, \rho, \mathrm{c} \in \mathbb{C}$, $\sigma \in (0,1) \cup N$, $\mathrm{Re}(\lambda) > -\mathrm{Re}(\alpha + v)$,

$\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$ and $k > 0$, $\mathrm{p} \in \boldsymbol{R}_0^+$,

we have following left fractional formula.

$$\left(D_{\alpha,v}^{+}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c}}(\mathrm{t};p)\right]\right)(x)$$

$$= x^{\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)}\frac{x^n}{n!}\frac{\Gamma\left(\alpha+v+\lambda+n\right)}{\Gamma\left(\alpha+\lambda+\mathrm{n}\right)}$$

…(4.5.9)

**Corollary-10**

For $\alpha, v, \lambda, l, m, \rho, \mathrm{c} \in \mathbb{C}$, $\sigma \in (0,1) \cup N$, $R(v) > 0$,

$\mathrm{Re}(\lambda) < \mathrm{Re}(\alpha + v) - [\mathrm{Re}(v)]$, $\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$ and

$k > 0$, $\mathrm{p} \in \boldsymbol{R}_0^+$,

we have the following right fractional formula.

$$\left(D_{\alpha,v}^{-}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c}}\left(\frac{1}{t};p\right)\right]\right)(x)$$

$$= x^{\lambda-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(nl+m\right)n!}\left(\frac{1}{x}\right)^n\frac{\Gamma\left(\alpha+v-\lambda+1+n\right)}{\Gamma\left(\alpha-\lambda+1+\mathrm{n}\right)}$$

…(4.5.10)

At $\mu = -v$ the result (4.5.7) and (4.5.8) reduce to the Riemann-Liouville fractional derivative of extended k-generalized Mittag-Leffler function. These are given as below in Corollary-11 and Corollary-12.

**Corollary-11**

For $v, \lambda, l, m, \rho, \mathrm{c} \in \mathbf{C}$, $\sigma \in (0,1) \cup N$, $\operatorname{Re}(v) > 0, \operatorname{Re}(\lambda) > 0,$

$\operatorname{Re}(l) > 0, \operatorname{Re}(m) > 0, \operatorname{Re}(\rho) > 0, \operatorname{Re}(c) > 0$ and $k > 0$, $\mathrm{p} \in \boldsymbol{R}_0^+$,

we have the following left fractional formula.

$$\left(D_{0+}^{v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c}(\mathrm{t};p)\right]\right)(x)$$

$$= x^{-v+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)}\frac{x^n}{n!}\frac{\Gamma(\lambda+n)}{\Gamma(-v+\lambda+\mathrm{n})} \qquad \ldots(4.5.11)$$

**Corollary-12**

For $v, \lambda, l, m, \rho, \mathrm{c} \in \mathbf{C}$, $\sigma \in (0,1) \cup N$,

$\operatorname{Re}(l) > 0, \operatorname{Re}(m) > 0, \operatorname{Re}(\rho) > 0, \operatorname{Re}(c) > 0, \operatorname{Re}(v) > 0,$

$\operatorname{Re}(\lambda) < \operatorname{Re}(v) - \left[\operatorname{Re}(v)\right]$ and $k > 0$, $\mathrm{p} \in \boldsymbol{R}_0^+$,

we have the following right fractional formulae.

$$\left(D_{-}^{v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c}}\left(\frac{1}{t};p\right)\right]\right)(x)$$

$$= x^{-v+\lambda-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(nl+m)n!}\left(\frac{1}{x}\right)^n\frac{\Gamma(v-\lambda+1+n)}{\Gamma(-\lambda+1+\mathrm{n})} \qquad \ldots(4.5.12)$$

(iii) Application of results (4.4.5) and (4.4.6) : we obtain certain known fractional derivatives. If in results (4.4.5) and (4.4.6) we take $\sigma = 1$ and $k = 1$, then in view of the definition of Fox-wright function [92], these reduce to the known results due to Araci et al [3, p.9, eq. (36); p.10, eq. (37)].

## SECTION-B

**4.6** Recently Jain et al. [45] studied extended k-generalized Mittag-Leffler function $E_{k,l,m}^{\rho,\sigma;c}(x;p)$ and Bin-Saad and Younis [5] introduced and studied a new extended Mittag-Leffler type function of arbitrary order $E_{\alpha,\beta}^{\rho,c;j,\gamma}(x;p)$. To unify the above related Mittag-Leffler functions we considered here the extended k-generalized Mittag-Leffler type function of arbitrary order $E_{k,l,m}^{\rho,\sigma;c;j,\gamma}(x;p)$. In this section we have established six theorems on fractional integral and derivatives formulae for this extended k-generalized Mittag-Leffler type function of arbitrary order.

As applications, our main theorems established in this section fractional integral and differential formulae related to Riemann-Liouville, Erdelyi-Kober and Saigo fractional operators are obtained. This extended k-generalized Mittag-Leffler type function is reducible in to four parameter Mittag-Leffler function therefore the certain known fractional differentiation and integration formulae are shown to be derived.

## 4.7 THE MITTAG-LEFFLER TYPE FUNCTIONS OF ARBITRARY ORDER

To unify the $j$-pseudo-hyperbolic, $j$-pseudo trigonometric and $\lambda$-hyperbolic function, Pathan and Bin-Saad [75] introduced and studied the following Mittag-Leffler type function of arbitrary order (see also [91]):

$$E_{v,\mu}^{j,k}(z)=\sum_{n=0}^{\infty}\frac{z^{nj+k}}{\Gamma\left(\mu+v\left(nj+k\right)\right)} \qquad \ldots(4.7.1)$$

where $j\geq 1, k\geq 0, v,\mu,z\in C; \mathrm{Re}(v)>0, \mathrm{Re}(\mu)>0.$

A new extended Mittag-Leffler type function of arbitrary order is studied by Bin-Saad and Younis [5] . It is represented and defined as follows [5, p.5, eq. (24)]:

$$E_{\alpha,\beta}^{\rho,c;j,\gamma}(z;p)=\sum_{n=0}^{\infty}\frac{B\left(\rho+n,c-\rho;p\right)(c)_n}{n!B\left(\rho,c-\rho\right)\Gamma\left(\beta+\alpha\left(nj+\gamma\right)\right)}z^{nj+\gamma} \qquad \ldots(4.7.2)$$

Where $\alpha,\beta,\gamma,c\in C,$

$\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(c)>0;\ j\geq 1,k\geq 0,\ \ p\in R_0^+$ and $B(x,y;p)$ is an extension of Beta function defined in [9] as follows:

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}\,e^{-\frac{p}{t(1-t)}}dt \qquad \dots(4.7.3)$$

$\mathrm{Re}(x)>0,\mathrm{Re}(y)>0,\mathrm{Re}(p)>0$ and $B(x,y;0)=B(x,y)$ in the familiar beta function (see [95]). It is clear that at $k=1$, extended $k$-Beta function defined in (4.2.36) coincide with eq. (4.7.3).

To unify the Mittag-Leffler function and all the related functions, we consider here the following $k$-generalized Mittag-Leffler type function of arbitrary order. It is represented and defined as follows.

$$E_{k,l,m}^{\rho,\sigma;c;j,\gamma}(x;p)=\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{n\sigma,k}}{\Gamma_k(m+l(nj+\gamma))}x^{nj+\gamma} \qquad \dots(4.7.4)$$

$k>0;x,l,m,\rho\in C,\mathrm{Re}(l)>0,\mathrm{Re}(m)>0,\mathrm{Re}(\rho)>0,\mathrm{Re}(c)>0;\ j\geq 1,\gamma\geq 0,\ p\in R_0^+$

where extended $k$-Beta function $B_k(l,m;p)$ is defined in (4.2.36).

## 4.8 MAIN THEOREMS

**Marichev-Saigo-Maeda Fractional Integrals of the Function $E_{k,l,m}^{\rho,\sigma;c;j,\gamma}(t;p)$**

Two fractional integrals of the generalized k- extended Mittag-Leffler type function are established here by applying left and right sided Marichev-Saigo-Maeda fractional integral operators defined in (4.2.5) and (4.2.6). These image formulae are given in Theorem-1 and Theorem-2 as below.

**Theorem-1**

For $v,v',\mu,\mu',\eta,\lambda,t,l,m,\mathrm{c}\in\mathrm{C}$ with

$\mathrm{Re}(\lambda)>\max\{0,\mathrm{Re}(v+v'+\mu-\eta),\mathrm{Re}(v'-\mu')\}>0$,

$\mathrm{Re}(\eta)>0, \mathrm{Re}(c)>0, k>0, \mathrm{Re}(l)>0, \mathrm{Re}(m)>0, \mathrm{Re}(\rho)>0$ , $\sigma \in (0,1) \cup N; \mathrm{p} \in R_0^+$,

and $j \geq 1, \gamma \geq 0$,

we have the following Marichev-Saigo-Maeda left fractional integral,

$$\left(I_{0+}^{\nu,\nu',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c,j,\gamma}(t;p)\right]\right)(x)=x^{-\nu-\nu'+\eta+\lambda+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{\Gamma(\lambda+\gamma+nj)\Gamma(-\nu'+\mu'+\lambda+\gamma+nj)\Gamma(-\nu-\nu'-\mu+\eta+\lambda+\gamma+nj)}{\Gamma(\lambda+\mu'+\gamma+nj)\Gamma(-\nu-\nu'+\eta+\lambda+\gamma+nj)\Gamma(-\nu'-\mu+\eta+\lambda+\gamma+nj)}\frac{(x^j)^n}{n!}$$

…(4.8.1)

**Theorem-2**

For $\nu,\nu',\mu,\mu',\eta,\lambda,\rho,l,m,\mathrm{c}\in\mathbb{C}$ with

$\mathrm{Re}(\lambda)<1+\min\{\mathrm{Re}(-\mu),\mathrm{Re}(\nu+\nu'-\eta),\mathrm{Re}(\nu+\mu'-\eta)\}$,

$\mathrm{Re}(\eta)>0, \mathrm{Re}(l)>0, \mathrm{Re}(m)>0, \mathrm{Re}(\rho)>0, \mathrm{Re}(c)>0$ and $k>0, \mathrm{p}\in R_0^+$ and

$j\geq 1, \gamma\geq 0$,

we have the following Marichev-Saigo-Maeda right fractional integral,

$$\left(I_{-}^{\nu,\nu',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c,j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x)=x^{-\nu-\nu'+\eta+\lambda-\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{\Gamma(-\mu-\lambda+\gamma+nj+1)\Gamma(\nu+\nu'-\eta-\lambda+\gamma+nj+1)\Gamma(\nu+\mu'-\eta-\lambda+\gamma+nj+1)}{\Gamma(-\lambda+\gamma+nj+1)\Gamma(\nu-\mu-\lambda+\gamma+nj+1)\Gamma(\nu+\nu'+\mu'-\eta-\lambda+\gamma+nj+1)}\frac{(x^{-j})^n}{n!}$$

…(4.8.2)

**Marichev-Saigo-Maeda Fractional Derivative of the Function** $E_{k,l,m}^{\rho,\sigma;c;j,\gamma}(t;p)$

The two fractional derivative image formulae for the generalized k- extended Mittag-Leffler type function are obtained here by applying the left and right Marichev-Saigo-Maeda fractional derivative operators in view of (4.2.12) and (4.2.13). These formulae are given in the following Theorem-3 and Theorem-4 as below.

**Theorem-3**

For $v, v', \mu, \mu', \xi, \lambda, l, m, \rho, \mathrm{c} \in \mathbb{C}$ with

$\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(-v-v'-\mu'+\xi), \mathrm{Re}(\mu-v)\}$,

$\mathrm{Re}(\xi) > 0, \mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$ and $k > 0$, $\mathrm{p} \in \boldsymbol{R}_0^+$,

$\sigma \in (0,1) \cup N$, and $j \geq 1, \gamma \geq 0$, we have the left Marichev-Saigo-Maeda left fractional derivative.

$$\left(D_{0+}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}(\mathrm{t};p)\right]\right)(x) = x^{\lambda+v+v'-\xi+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{(x^j)^n}{n!}\frac{\Gamma(\lambda+\gamma+nj)\Gamma(v-\mu+\lambda+\gamma+nj)\Gamma(v+v'+\mu'-\xi+\lambda+\gamma+nj)}{\Gamma(-\mu+\lambda+\gamma+nj)\Gamma(v+v'-\xi+\lambda+\gamma+nj)\Gamma(v+\mu'-\xi+\lambda+\gamma+nj)}$$

…(4.8.3)

**Theorem-4**

For $v, v', \mu, \mu', \xi, \lambda, l, m, \rho, \mathrm{c} \in \mathbb{C}$, $\sigma \in (0,1) \cup N$

$\mathrm{Re}(\lambda) < 1 + \min\{\mathrm{Re}(\mu'), \mathrm{Re}(-v-v'+\xi), \mathrm{Re}(-v'-\mu+\xi)\}$,

$\mathrm{Re}(\xi) > 0, \mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$ and $k > 0, \mathrm{p} \in \boldsymbol{R}_0^+$ and

$j \geq 1, \gamma \geq 0$,

we have the Marichev-Saigo-Maeda right fractional derivative,

$$\left(D_{-}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x) = x^{v+v'-\xi+\lambda-\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{(x^{-j})^n}{n!}\frac{\Gamma(\mu'-\lambda+\gamma+nj+1)\Gamma(-v-v'+\xi-\lambda+\gamma+nj+1)\Gamma(-v'-\mu+\xi-\lambda+\gamma+nj+1)}{\Gamma(-\lambda+\gamma+nj+1)\Gamma(-v'+\mu'-\lambda+\gamma+nj+1)\Gamma(-v-v'-\mu+\xi-\lambda+\gamma+nj+1)}$$

…(4.8.4)

**Caputo-Type Marichev-Saigo-Maeda Fractional Derivative of the Function $E_{k,l,m}^{\rho,\sigma;c;j,\gamma}(t;p)$**

In the next two Theorem-5 and Theorem-6 the two fractional derivatives of the generalized k- extended Mittag-Leffler type function are established by applying left and right sided Caputo-type Marichev-Saigo-Maeda fractional derivatives operators defined in (4.2.26) and (4.2.27). The Theorem-5 and Theorem-6 are given as below.

**Theorem-5**

For $v, v', \mu, \mu', \xi, \lambda, l, m, \rho, \mathrm{c} \in \mathbf{C}$, $\sigma \in (0,1) \cup N$,

$\mathrm{Re}(\lambda+\gamma) - b > \max\{0, \mathrm{Re}(-v-v'-\mu'+\xi), \mathrm{Re}(\mu-v)\}$, $b = [\mathrm{Re}(\xi)]+1$,

$\mathrm{Re}(\xi)>0, \mathrm{Re}(l)>0, \mathrm{Re}(m)>0, \mathrm{Re}(\rho)>0, \mathrm{Re}(c)>0$ and $k>0$, $\mathrm{p} \in \boldsymbol{R}_0^+$, and

$j \geq 1, \gamma \geq 0$,

we have the left Caputo-type Marichev-Saigo-Maeda fractional derivative as follows.

$$\left({}^{C}D_{0+}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}(\mathrm{t};p)\right]\right)(x) = x^{v+v'-\xi+\lambda+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{(x^j)^n}{n!}\frac{\Gamma(\lambda+\gamma+nj)\Gamma(v-\mu+\lambda-b+\gamma+nj)\Gamma(v+v'+\mu'-\xi+\lambda-b+\gamma+nj)}{\Gamma(-\mu+\lambda-b+\gamma+nj)\Gamma(v+v'-\xi+\lambda+\gamma+nj)\Gamma(v+\mu'-\xi+\lambda-b+\gamma+nj)}$$

…(4.8.5)

**Theorem-6**

For $v, v', \mu, \mu', \xi, \lambda, l, m, \rho, \mathrm{c} \in \mathbf{C}$, $\sigma \in (0,1) \cup N$,

$\mathrm{Re}(\lambda+\gamma) + b > \max\{-\mathrm{Re}(\mu'), \mathrm{Re}(v'+\mu-\xi), \mathrm{Re}(v+v'-\xi)+b\}$,

$\mathrm{b} = [\mathrm{Re}(\xi)]+1, \mathrm{Re}(l)>0, \mathrm{Re}(m)>0, \mathrm{Re}(\rho)>0, \mathrm{Re}(c)>0$ and $k>0, \mathrm{p} \in \boldsymbol{R}_0^+$

and $j \geq 1, \gamma \geq 0$ we have the right Caputo-type Marichev-Saigo-Maeda fractional derivative as follows:

$$\left({}^{C}D_{-}^{v,v',\mu,\mu',\xi}\left[t^{-\lambda}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x) = x^{v+v'-\xi-\lambda-\gamma}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{(x^{-j})^n}{n!}\frac{\Gamma(\mu'+\lambda+b+\gamma+nj)\Gamma(-v-v'+\xi+\lambda+\gamma+nj)\Gamma(-v'-\mu+\xi+\lambda+b+\gamma+nj)}{\Gamma(\lambda+\gamma+nj)\Gamma(-v'+\mu'+\lambda+b+\gamma+nj)\Gamma(-v-v'-\mu+\xi+\lambda+b+\gamma+nj)}$$

…(4.8.6)

**OUTLINE OF PROOFS**

**Proof of (4.8.1)**

Let the left hand side of (4.8.1) is denoted by $\Delta_1$, i.e.

$$\left(I_{0+}^{v,v',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c,j,\gamma}(t;p)\right]\right)(x)$$

Now on using the definition given in (4.7.4) and then on arranging the order of summation and integration we have

$$\Delta_1=\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho,c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)\ n!}\left(I_{0+}^{v,v',\mu,\mu',\eta}t^{\lambda+\gamma+nj-1}\right)(x)$$

Now on applying the formulae (4.2.20), we get

$$\Delta_1=\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho,c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)\ n!}$$

$$\frac{\Gamma\left(\lambda+\gamma+nj\right)\Gamma\left(-v'+\mu'+\lambda+\gamma+nj\right)\Gamma\left(-v-v'-\mu+\eta+\lambda+\gamma+nj\right)}{\Gamma\left(\mu'+\lambda+\gamma+nj\right)\Gamma\left(-v-v'+\eta+\lambda+\gamma+nj\right)\Gamma\left(-v'-\mu+\eta+\lambda+\gamma+nj\right)}\ x^{-v-v'+\eta+\lambda+nj+\gamma-1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result (4.8.1).

**Proof of (4.8.2)**

Let the left hand side of (4.8.2) is denoted by $\Delta_2$, i.e.

$$\Delta_2=\left(I_{-}^{v,v',\mu,\mu',\eta}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x)$$

Now on using the definition given in (4.7.4) and then on arranging the order of summation and integration we have

$$\Delta_2=\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho,c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)n!}\left(I_{-}^{v,v',\mu,\mu',\eta}t^{\lambda-\gamma-nj-1}\right)(x)$$

Now on applying the formulae (4.2.21), we get

$$\Delta_2=\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho,c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)\mathrm{n}!}\frac{\Gamma\left(-\mu-\lambda+\gamma+nj+1\right)}{\Gamma\left(-\lambda+\gamma+nj+1\right)}$$

$$\frac{\Gamma\left(v+v'-\eta-\lambda+\gamma+nj+1\right)\Gamma\left(v+\mu'-\eta-\lambda+\gamma+nj+1\right)}{\Gamma\left(v-\mu-\lambda+\gamma+nj+1\right)\Gamma\left(v+v'+\mu'-\eta-\lambda+\gamma+nj+1\right)}x^{\lambda-\gamma-nj-v-v'+\eta-1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result (4.8.2).

**Proof of (4.8.3)**

Let the left hand side of (4.8.3) is denoted by $\Delta_3$, i.e.

$$\Delta_3 = \left( D_{0+}^{v,v',\mu,\mu',\xi} \left[ t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma} (\mathrm{t}; p) \right] \right)(x)$$

Now on using the definition (4.7.4) and applying the left Marichev-Saigo-Maeda left fractional derivative and on changing the order of differentiation and summation, we have

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{B_k \left( \rho + n\sigma k, \mathrm{c} - \rho; \mathrm{p} \right) (c)_{n\sigma,k}}{B_k \left( \rho, c - \rho \right) \Gamma_k \left( m + l(\gamma + nj) \right) n!} \left( D_{0+}^{v,v',\mu,\mu',\xi} t^{\lambda + nj + \gamma - 1} \right)(x)$$

On applying the formulae (4.2.22), it takes the following form

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{B_k \left( \rho + n\sigma k, \mathrm{c} - \rho; \mathrm{p} \right) (c)_{n\sigma,k}}{B_k \left( \rho, c - \rho \right) \Gamma_k \left( m + l(\gamma + nj) \right) n!}$$

$$\frac{\Gamma(\lambda + \gamma + nj) \Gamma(v + v' + \mu' - \xi + \lambda + \gamma + nj) \Gamma(v - \mu + \lambda + \gamma + nj)}{\Gamma(-\mu + \lambda + \gamma + nj) \Gamma(v + v' - \xi + \lambda + \gamma + nj) \Gamma(v + \mu' - \xi + \lambda + \gamma + nj)} x^{v + v' - \xi + \lambda + nj + \gamma - 1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result (4.8.3).

**Proof of (4.8.4)**

Let the left hand side of (4.8.4) is denoted by $\Delta_4$ i.e.,

$$\Delta_4 = \left( D_{-}^{v,v',\mu,\mu',\xi} \left[ t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma} \left( \frac{1}{t}; p \right) \right] \right)(x)$$

Now on using the definition (4.7.4) and applying the right Marichev-Saigo-Maeda fractional derivative and on changing the order of differentiation and summation, we get

$$\Delta_4 = \sum_{n=0}^{\infty} \frac{B_k \left( \rho + n\sigma k, \mathrm{c} - \rho; \mathrm{p} \right) (c)_{n\sigma,k}}{B_k \left( \rho, c - \rho \right) \Gamma_k \left( m + l(\gamma + nj) \right)} \left( D_{-}^{v,v',\mu,\mu',\xi} t^{\lambda - nj - \gamma - 1} \right)(x)$$

On applying the formulae (4.2.23), it takes the following form

$$\Delta_4=\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho,c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)n!}\frac{\Gamma\left(\mu'-\lambda+\gamma+nj+1\right)}{\Gamma\left(-\lambda+\gamma+nj+1\right)}$$

$$\frac{\Gamma\left(-v-v'+\xi-\lambda+\gamma+nj+1\right)\Gamma\left(-v'-\mu+\xi-\lambda+\gamma+nj+1\right)}{\Gamma\left(-v'+\mu'-\lambda+\gamma+nj+1\right)\Gamma\left(-v-v'-\mu+\xi-\lambda+\gamma+nj+1\right)}\ x^{\lambda+v+v'+\xi-nj-\gamma-1}$$

On using (4.2.35) and (4.2.39), we at once arrive at the desired result (4.8.4).

**Proof of (4.8.5)**

Let the left hand side of (4.8.5) is denoted by $\Delta_5$ i.e.

$$\Delta_5=\left({}^{C}D_{0+}^{v,v',\mu,\mu',\xi}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\mathrm{t};p\right)\right]\right)(x)$$

Now using the definition (4.7.4) and on changing the order of differentiation and summation, we have

$$\Delta_5=\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho,c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)}\left({}^{C}D_{0+}^{v,v',\mu,\mu',\xi}t^{\lambda+\gamma+nj-1}\right)(x)$$

On using the formulae (4.2.28), it can be written as

$$\Delta_5=\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho,c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)}$$

$$\frac{\Gamma\left(\lambda+\gamma+nj\right)\Gamma\left(v-\mu+\lambda-b+\gamma+nj\right)}{\Gamma\left(-\mu+\lambda-b+\gamma+nj\right)\Gamma\left(v+v'-\xi+\lambda+\gamma+nj\right)}$$

$$\frac{\Gamma\left(v+v'+\mu'-\xi+\lambda-b+\gamma+nj\right)}{\Gamma\left(v+\mu'-\xi+\lambda-b+\gamma+nj\right)}\ x^{\lambda+\gamma+nj+v+v'-\xi-1}$$

On using (4.2.35) and (4.2.39), we at once get the required result (4.8.5)

**Proof of (4.8.6)**

Let the left hand side of (4.8.6) is denoted by $\Delta_6$ i.e.,

$$\Delta_6=\left({}^{C}D_{-}^{v,v',\mu,\mu',\xi}\left[t^{-\lambda}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x)$$

Now using the definition (4.7.4) and on changing the order of differentiation and summation, we get

$$\Delta_6 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho; \mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)} \left({}^{C}D_{-}^{v,v',\mu,\mu',\xi} t^{-(\lambda+nj+\gamma)}\right)(x)$$

On using the formulae (4.2.29) it can be written as

$$\Delta_6 = \sum_{n=0}^{\infty} \frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho; \mathrm{p}\right)(c)_{n\sigma,k}}{B_k\left(\rho, c-\rho\right)\Gamma_k\left(m+l\left(\gamma+nj\right)\right)}$$

$$\frac{\Gamma\left(\mu'+\lambda+b+\gamma+nj\right)\Gamma\left(-v-v'+\xi+\lambda+\gamma+nj\right)}{\Gamma\left(\lambda+\gamma+nj\right)\Gamma\left(-v'+\mu'+\lambda+b+\gamma+nj\right)}$$

$$\frac{\Gamma\left(-v'-\mu+\xi+\lambda+b+\gamma+nj\right)}{\Gamma\left(-v-v'-\mu+\xi+\lambda+b+\gamma+nj\right)} \; x^{v+v'-\xi-\lambda-nj-\gamma}$$

On using (4.2.35) and (4.2.39), we at once get the required result (4.8.6).

## 4.9 APPLICATIONS

As application of the main results the corresponding formulae for Saigo operators, Erdelyi-Kober operators and Riemann-Liouville operators are obtained and given as following corollaries.

(i) Application of the results (4.8.1) and (4.8.2): we obtain the following fractional integral formulae for Saigo fractional integral operator.

**Corollary-1**

For $v, \mu, t, l, m, \mathrm{c}, \rho, \lambda, \alpha \in \mathbb{C}$ , $\mathrm{Re}(v)>0, \mathrm{Re}(m)>0, \mathrm{Re}(\rho)>0$ ,

$\sigma \in (0,1) \cup N; k>0, \mathrm{p} \in R_0^{+}$ with $\mathrm{Re}(\lambda) > \max\{0, \mathrm{Re}(\mu-\alpha)\}$, and $j \geq 1, \gamma \geq 0$,

we have the following left fractional integral formula.

$$\left(I_{0+}^{v,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}(\mathrm{t};p)\right]\right)(x) = x^{-\mu+\lambda+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho; \mathrm{p}\right)}{\Gamma_k(\rho)\Gamma_k(c-\rho)}$$

$$\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(m+l\left(\gamma+nj\right)\right)} \frac{\Gamma\left(\lambda+\gamma+nj\right)\Gamma\left(-\mu+\alpha+\lambda+\gamma+nj\right)}{\Gamma\left(-\mu+\lambda+\gamma+nj\right)\Gamma\left(v+\alpha+\lambda+\gamma+nj\right)} \frac{\left(x^j\right)^n}{n!}$$

…(4.9.1)

**Proof of (4.9.1)**

The result (4.9.1) is easily established following similar lines as in the proof of result (4.8.1) in view of (4.2.9).

**Corollary-2**

For $v, \mu, \alpha, \lambda, l, m, \rho, \mathrm{c} \in \mathrm{C}$,

$\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0, \mathrm{Re}(v) > 0,\ \sigma \in (0,1) \cup N$

$\mathrm{Re}(\lambda) < 1 + \min\{\mathrm{Re}(\mu), \mathrm{Re}(\alpha)\}\quad k > 0, \mathrm{p} \in R_0^+$, and $j \geq 1, \gamma \geq 0$,

we have the following right fractional integral formula.

$$\left(I_{-}^{v,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c,j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x) = x^{-\mu+\lambda-\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}$$

$$\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}\frac{\Gamma(\alpha-\lambda+\gamma+nj+1)\Gamma(\mu-\lambda+\gamma+nj+1)}{\Gamma(-\lambda+\gamma+nj+1)\Gamma(v+\mu+\alpha-\lambda+\gamma+nj+1)}\frac{(x^{-j})^n}{n!}$$

…(4.9.2)

**Proof of (4.9.2)**

The result (4.9.2) is easily established following similar lines as in the proof of result (4.8.2) in view of (4.2.10).

The results in (4.9.1) and (4.9.2) in turn at $\mu = 0$ provide the following fractional integrals related to Erdelyi-Kober operators defined in (4.2.16) and (4.2.17):

**Corollary-3**

For $v, \mu, t, l, m, \mathrm{c}, \rho, \lambda, \alpha \in \mathrm{C}$, $\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0,$

$\mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0, \mathrm{Re}(v) > 0$, $\sigma \in (0,1) \cup N; \mathrm{p} \in R_0^+$ and $k > 0$,

$\mathrm{Re}(\lambda) > -\mathrm{Re}(\alpha)$, and $j \geq 1, \gamma \geq 0$,

we have the following left fractional integral formula.

$$\left(I_{\alpha,v}^{+}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}(t;p)\right]\right)(x)$$

$$= x^{\lambda+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}\frac{\Gamma(\alpha+\lambda+\gamma+nj)}{\Gamma(v+\alpha+\lambda+\gamma+nj)}\frac{(x^j)^n}{n!}$$

…(4.9.3)

**Corollary-4**

For $v, \alpha, \lambda, l, m, \rho, \mathrm{c} \in \mathbb{C}$ $\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(v) > 0,$ $\sigma \in (0,1) \cup N$ $\mathrm{Re}(\lambda) < 1 + \mathrm{Re}(\alpha)$ $k > 0, \mathrm{p} \in R_0^+$ ,and $j \geq 1, \gamma \geq 0$ , we have the following right fractional integral formula.

$$\left(K_{\alpha,v}^{-}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x)$$

$$= x^{\lambda-\gamma-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(m+l\left(\gamma+nj\right)\right)}\frac{\Gamma\left(\alpha-\lambda+\gamma+nj+1\right)}{\Gamma\left(\alpha+v-\lambda+\gamma+nj+1\right)}\frac{\left(x^{-j}\right)^n}{n!}$$

…(4.9.4)

At $\mu = -v$ the result (4.9.1) and (4.9.2) reduce to the Riemann-Liouville fractional integral of extended k-generalized Mittag-Leffler type function. These are given as below in Corollary-5 and Corollary-6.

**Corollary-5**

For $t, v, \lambda, \mathrm{c}, l, m, \rho \in \mathbb{C},$ $\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0,$ $\mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0, \mathrm{Re}(v) > 0$ , $\sigma \in (0,1) \cup N;$ $\mathrm{p} \in R_0^+$ and $k > 0$ $\mathrm{Re}(\lambda) > 0$, and $j \geq 1, \gamma \geq 0$ ,

we have the following left fractional integral formulae

$$\left(I_{0+}^{v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(t;p\right)\right]\right)(x)$$

$$= x^{v+\lambda+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(m+l\left(\gamma+nj\right)\right)}\frac{\Gamma\left(\lambda+\gamma+nj\right)}{\Gamma\left(v+\lambda+\gamma+nj\right)}\frac{\left(x^{j}\right)^n}{n!}$$

…(4.9.5)

**Corollary-6**

For $v, \lambda, l, m, \rho, \mathrm{c} \in \mathbb{C}$, $\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0, \mathrm{Re}(v) > 0,$ $\sigma \in (0,1) \cup N$ $\mathrm{Re}(v) < 1 - \mathrm{Re}(\lambda),$ $k > 0, \mathrm{p} \in R_0^+$ and $j \geq 1, \gamma \geq 0$ ,we have the following right fractional integral formula.

$$\left(I_{-}^{v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x)$$

$$= x^{v+\lambda-\gamma-1}\sum_{n=0}^{\infty}\frac{B_k\left(\rho+n\sigma k, \mathrm{c}-\rho;\mathrm{p}\right)}{\Gamma_k\left(\rho\right)\Gamma_k\left(c-\rho\right)}\frac{\Gamma_k\left(c+n\sigma k\right)}{\Gamma_k\left(m+l\left(\gamma+nj\right)\right)}\frac{\Gamma\left(-v-\lambda+\gamma+nj+1\right)}{\Gamma\left(-\lambda+\gamma+nj+1\right)}\frac{\left(x^{-j}\right)^n}{n!}$$

…(4.9.6)

(ii) Application of results (4.8.3) and (4.8.4): we obtain the following fractional derivative formulae for Saigo differential operator:

**Corollary-7**

For $v,\mu,\alpha,\lambda,l,m,\rho,\mathrm{c}\in\mathrm{C}$, $\sigma\in(0,1)\cup N$, $\mathrm{Re}(\lambda)>-\min\{0,\mathrm{Re}(v+\mu+\alpha)\}$,

$\mathrm{Re}(l)>0$, $\mathrm{Re}(m)>0, \mathrm{Re}(\rho)>0, \mathrm{Re}(c)>0$ and $k>0$, $\mathrm{p}\in \boldsymbol{R}_0^+$, and

$j\geq 1, \gamma\geq 0$,

we have the following left fractional formula.

$$\left(D_{0+}^{v,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c,j,\gamma}(\mathrm{t};p)\right]\right)(x)=x^{\mu+\lambda+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{(x^j)^n}{n!}\frac{\Gamma(\lambda+\gamma+nj)\Gamma(v+\mu+\alpha+\lambda+\gamma+nj)}{\Gamma(\alpha+\lambda+\gamma+nj)\Gamma(\mu+\lambda+\gamma+nj)} \qquad \ldots(4.9.7)$$

**Proof of (4.9.7)**

The result (4.9.7) is easily established following similar lines as in the proof of result (4.8.3) in view of (4.2.14).

**Corollary-8**

For $v,\mu,\alpha,\lambda,l,m,\rho,\mathrm{c}\in\mathrm{C}$, $\sigma\in(0,1)\cup N$,

$\mathrm{Re}(\lambda-\gamma)<1+\min\{\mathrm{Re}(-\mu)-b,\ \mathrm{Re}(v+\alpha)\}, b=[\mathrm{Re}(v)]+1$,

$\mathrm{Re}(l)>0, \mathrm{Re}(m)>0, \mathrm{Re}(\rho)>0, \mathrm{Re}(c)>0$ and $k>0$, $\mathrm{p}\in \boldsymbol{R}_0^+$ and

$j\geq 1, \gamma\geq 0$,

we have the following right fractional formula.

$$\left(D_{-}^{v,\mu,\alpha}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c,j,\gamma}(\mathrm{t};p)\right]\right)(x)=x^{\mu+\lambda-\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}$$

$$\frac{(x^{-j})^n}{n!}\frac{\Gamma(-\mu-\lambda+1+\gamma+nj)\Gamma(v+\alpha-\lambda+1+\gamma+nj)}{\Gamma(-\lambda+1+\gamma+nj)\Gamma(-\mu+\alpha-\lambda+1+\gamma+nj)} \qquad \ldots(4.9.8)$$

**Proof of (4.9.8)**

The result (4.9.8) is easily established following similar lines as in the proof result (4.8.4) in view of (4.2.15).

The results (4.9.7) and (4.9.8) in turn at $\mu = 0$ provides the following fractional derivatives related to Erdelyi-Kober operator defined in (4.2.18) and (4.2.19):

**Corollary-9**

For $\alpha, v, \lambda, l, m, \rho, \mathrm{c} \in \mathrm{C}$, $\sigma \in (0,1) \cup N$, $\mathrm{Re}(\lambda) > -\mathrm{Re}(\rho + v)$, $\mathrm{Re}(l) > 0$,

$\mathrm{Re}(m) > 0$, $\mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$ and $k > 0$, $\mathrm{p} \in \boldsymbol{R}_0^+$ and $j \geq 1, \gamma \geq 0$,

we have following left fractional formula.

$$\left(D_{\alpha,v}^{+}\left[t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}(\mathrm{t}; p)\right]\right)(x)$$

$$= x^{\lambda+\gamma-1} \sum_{n=0}^{\infty} \frac{B_k(\rho + n\sigma k, \mathrm{c} - \rho; \mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)} \frac{\Gamma_k(c + n\sigma k)}{\Gamma_k(m + l(\gamma + nj))} \frac{\Gamma(\alpha + v + \lambda + \gamma + nj)}{\Gamma(\alpha + \lambda + \gamma + nj)} \frac{(x^j)^n}{n!}$$

…(4.9.9)

**Corollary-10**

For $\alpha, v, \lambda, l, m, \rho, \mathrm{c} \in \mathrm{C}$, $\sigma \in (0,1) \cup N$, $R(v) > 0$,

$\mathrm{Re}(\lambda - \gamma) < \mathrm{Re}(\alpha + v) - [\mathrm{Re}(v)]$, $\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0$

and $k > 0$, $\mathrm{p} \in \boldsymbol{R}_0^+$ and $j \geq 1, \gamma \geq 0$,we have the following right fractional formula.

$$\left(D_{\alpha,v}^{-}\left[t^{\lambda-1} E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t}; p\right)\right]\right)(x)$$

$$= x^{\lambda-\gamma-1} \sum_{n=0}^{\infty} \frac{B_k(\rho + n\sigma k, \mathrm{c} - \rho; \mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)} \frac{\Gamma_k(c + n\sigma k)}{\Gamma_k(m + l(\gamma + nj))} \frac{\Gamma(\alpha + v - \lambda + 1 + \gamma + nj)}{\Gamma(\alpha - \lambda + 1 + \gamma + nj)} \frac{(x^{-j})^n}{n!}$$

…(4.9.10)

At $\mu = -v$, result (4.9.7) and (4.9.8) reduce to the Riemann-Liouville fractional derivative of extended k-generalized Mittag-Leffler type function. These are given as below in Corollary-11 and Corollary-12.

**Corollary-11**

For $v, \lambda, l, m, \rho, \mathrm{c} \in \mathbf{C}$, $\sigma \in (0,1) \cup N$ $\mathrm{Re}(v) > 0, \mathrm{Re}(\lambda) > 0,$

$\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(c) > 0$ $\mathrm{Re}(\rho) > 0$ and $k > 0$, $\mathbf{p} \in \boldsymbol{R}_0^+$ and

$j \geq 1, \gamma \geq 0$,

we have the following left fractional formula.

$$\left(D_{0+}^{v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;c,j,\gamma}(\mathrm{t};p)\right]\right)(x)$$

$$= x^{-v+\lambda+\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}\frac{\Gamma(\lambda+\gamma+nj)}{\Gamma(-v+\lambda+\gamma+nj)}\frac{(x^j)^n}{n!}$$

…(4.9.11)

**Corollary-12**

For $v, \lambda, l, m, \rho, \mathrm{c} \in \mathbf{C}$, $\sigma \in (0,1) \cup N$,

$\mathrm{Re}(l) > 0, \mathrm{Re}(m) > 0, \mathrm{Re}(\rho) > 0, \mathrm{Re}(c) > 0, \mathrm{Re}(v) > 0,$

$\mathrm{Re}(\lambda-\gamma) < \mathrm{Re}(v) - [\mathrm{Re}(v)]$ and $k > 0$, $\mathbf{p} \in \boldsymbol{R}_0^+$ and $j \geq 1, \gamma \geq 0$,

we have the following right fractional formula.

$$\left(D_{-}^{-v}\left[t^{\lambda-1}E_{k,l,m}^{\rho,\sigma;\mathrm{c},j,\gamma}\left(\frac{1}{t};p\right)\right]\right)(x)$$

$$= x^{-v+\lambda-\gamma-1}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,\mathrm{c}-\rho;\mathrm{p})}{\Gamma_k(\rho)\Gamma_k(c-\rho)}\frac{\Gamma_k(c+n\sigma k)}{\Gamma_k(m+l(\gamma+nj))}\frac{\Gamma(v-\lambda+1+\gamma+nj)}{\Gamma(-\lambda+1+\gamma+nj)}\frac{(x^{-j})^n}{n!}$$

…(4.9.12)

(iii) Application of results (4.8.5) and (4.8.6): If in results (4.8.5) and (4.8.6) we take $j=1,\ ,\gamma=0,\ k=1$ *and* $\sigma=1$, then in view of the definition of Fox-wright function [92], these reduce to known results due to Araci et al. [3, p.9, eq. (36); p.10, eq. (37)].

## 4.10 SPECIAL CASES

For particular value $\sigma = 1, j = 1$ *and* $\gamma = 0$ the extended k-generalized Mittag-Leffler type function reduce to a function studied by Rahman et.al.[79] which in turn at $p = 0$ provides the Mittag-Leffler function due to Dorrego and Cerutti[17], therefore the related Marichev-Saigo-Meada fractional integral and derivative formulae can be obtained for these simpler functions. Also by considering $k = 1$, $j = 1$ *and* $\gamma = 0$ the fractional integrals and derivatives involving the Mittag-Leffler function due to Prabhakar [78], Haubold et al. [35], and Mittal et al.[68] can also be derived.

It is important to emphasize that the results established in this chapter generalize the results studied recently by Pathan and Bin-Saad [75], Shahwan et al.[91], Araci et al. [3] and Bin-Saad and Younis [5].

# CHAPTER-5

# FRACTIONAL CALCULUS OPERATORS AND CERTAIN FRACTIONAL ORDER KINETIC EQUATIONS AND DIFFERENTIAL EQUATIONS WITH THEIR SOLUTIONS

The main results of this chapter have been published / accepted / communicated for publication as per details below:

1. The general hybrid-type fractional order kinetic equation involving general multivariable function for Riemann-Caputo fractional derivative operator
   (Communicated for publication)

2. Solution of general hybrid-type fractional order kinetic equations associated with the multivariable general function for Hilfer type fractional derivative operator
   (Communicated for publication)

## 5.1 INTRODUCTION

In this chapter, we have obtained the solution of a general family of hybrid-type fractional order kinetic equation for fractional derivative operators involving multivariable general function associated to the multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta functions. This chapter is divided into two sections; Section-A and B.

In section-A, we have find out the solution of general hybrid-type fractional order kinetic equation with the Liouville-Caputo fractional derivative operator ${}^{LC}D_{0+}^{\sigma}$ involving the multivariable general function. In Section-B, the solution of general hybrid-type fractional order kinetic equation with the Hilfer type fractional derivative operator ${}^{H}D_{0+}^{\sigma,\omega}$ involving the multivariable general function are obtained.

In the Section-A, we have established three theorems. The general hybrid-type fractional order kinetic equation involving the multivariable general function, multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta function respectively for the Liouville-Caputo fractional derivative operator ${}^{LC}D_{0+}^{\sigma}$ are solved here. Our results are very general and capable to be yielding solutions of many known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving Hurwitz-Lerch Zeta function of one and several variables as the applications.

In Section-B, three theorems are established in which the solution of the general hybrid-type fractional order kinetic equation involving the multivariable general function, multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta function respectively for the Hilfer type fractional derivative operator ${}^{H}D_{0+}^{\sigma,\omega}$ are investigated. As the applications, the kinetic equations considered in these theorems are capable to be yielding solutions of a significantly large number of known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving the Hurwitz-Lerch Zeta function of one and several variables.

## 5.2 DEFINITIONS AND PRILIMINARIES

The Hurwitz-Lerch Zeta function is defined and studied by Erdélyi et al. [19] in the following form see also ([95],[98]):

$$\phi(z;s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}\ , \qquad \ldots(5.2.1)$$

$(a\in C/Z_0^-, s\in \mathrm{C}$ when $|z|<1$ and $\mathrm{Re}(s)>1$ when $|z|=1)$

here $C$ represents the set of complex numbers, and $Z_0^-=Z^-\cup\{0\}=\{0,-1,-2,...\}$.

In 1997, Goyal and Laddha [28] defined and studied an extension of (5.2.1) in the following form:

$$\phi_\mu^*(z;s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(a+n)^s}\ , \qquad \ldots(5.2.2)$$

$(\mu\in \mathrm{C}; a\in C/Z_0^-; s\in \mathrm{C}$ when $|z|<1$ and $\mathrm{Re}(s-\mu)>1$ when $|z|=1)$

After words, many researchers studied many different generalization and extension of Hurwitz-Lerch Zeta function of one variable. The interested readers can refer to these publications for further researches. (Lin and Srivastava [58], Lin et al. [59], Choi et al [11], Gupta et al [21], Garg et. al. [23], Srivastava et al [107], Daman and Pathan [12], Srivastava et. al. [108] and Srivastava [104]).

Here $(\lambda)_n$ denotes the general pochhammer symbol, which is defined by Rainville [81].

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma\lambda}=\begin{cases}1, & n=0; \lambda\in C/\{0\}\\ \lambda(\lambda+1)....(\lambda+n-1), & n\in N; \lambda\in C\end{cases} \qquad \ldots(5.2.3)$$

A multivariable extension of the Hurwitz-Lerch Zeta function (5.2.2) defined by Raina and Chhajed [80]. We define this function at $b=0$ in the following manner:

$$\phi_{(\mu_i)}^{(p_i)}(x_1,...,x_n;\alpha,a)=\sum_{m_1,...,m_n=0}^{\infty}(a+\Omega)^{-\alpha}\prod_{i=1}^{n}\left\{\frac{(\mu_i)_{m_i}\,x_i^{m_i}}{m_i\,!}\right\}\ , \qquad \ldots(5.2.4)$$

$(\mu_i\geq 1\ (i=1,2,...,n); \mathrm{Re}(a)>0, \mathrm{Re}(p_i)>0;\ \max_{1\leq i\leq n}(|x_i|)\leq 1;\ \Omega=\sum_{i=1}^{n}p_i m_i)$

The further generalization of multivariable Hurwitz-Lerch Zeta function is studied by Gupta [30, p.114, eq. (2.7.3)] in the following form

$$\phi^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right),\left(\xi_k^{(i)}\right)}\left(z_1,...,z_r;s,a\right)=\sum_{n_1,...,n_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}\theta_i n_i\right)^s}\prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i!}\right\} \quad \ldots(5.2.5)$$

$\forall\ i=1,2,...,r;\ p_i,q_i\in N_o;\ \delta_j^{(i)}\in R^+,\ \mu_j^{(i)}\in C\left(i=1...p_i\right);\ a,\sigma_k^{(i)}\in C/Z_0^-;$

$\xi_k^{(i)}\in R^+\left(k=1...q_i\right);\ \min\left\{\operatorname{Re}(a),\operatorname{Re}(s)\right\}>0;\ \max_{1\le i\le r}|z_i|<1,$

for $i=1,2,...,r$, the array of parameters $\left(\mu_j^{(i)}\right)\equiv\left(\mu_1^{(i)}\right),...,\left(\mu_{p_i}^{(i)}\right)$ and similarly $\left(\sigma_k^{(i)}\right)\equiv\left(\sigma_1^{(i)}\right),...,\left(\sigma_{q_i}^{(i)}\right)$

A well known Mittag-Leffler function is defined and represented as follows [66] (see also [67],[113])

$$E_\alpha\left(z\right)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\alpha n+1\right)} \quad \ldots(5.2.6)$$

$(z\in \mathrm{C};\operatorname{Re}(\alpha)>0)$

A large number of generalization and extensions of the Mittag-Leffler function of single variable have been studied by several authors (Prabhakar [78], Srivastava and Tomovski [97], Shukla and Prajapati [93])

A multivariable extension of Mittag-Leffler function studied by Gautam [25] and Saxena et.al. [88] in the following form:

$$E^{(\gamma_r)(l_r)}_{(\rho_r),\lambda}\left(z_1,...,z_r\right)=\sum_{k_1,...,k_r=0}^{\infty}\frac{\left(\gamma_1\right)_{k_1l_1}...\left(\gamma_r\right)_{k_rl_r}}{\Gamma\left(\lambda+k_1\rho_1+...+k_r\rho_r\right)}\frac{z_1^{k_1}}{\left(k_1\right)!}...\frac{z_r^{k_r}}{\left(k_r\right)!} \quad \ldots(5.2.7)$$

where $\lambda,\gamma_j,l_j,\rho_j\in C;\ \operatorname{Re}\left(\rho_j\right)>0;\ \operatorname{Re}\left(l_j\right)>0,\lambda\notin Z_0,\ j=1,2,...,r$

Recently for suitably restricted sequence $\left\{\phi(n)\right\}_{n=0}^{\infty}$ Srivastava [94] introduced a class of function (see also [105],[110]) which is given by

$$\in_{\alpha,\beta}\left(\phi;z,s,a\right)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{\left(a+n\right)^s\Gamma\left(\alpha n+\beta\right)} \quad \ldots(5.2.8)$$

$\left(\alpha,\beta\in \mathrm{C};\operatorname{Re}(\alpha)>0\right)$

The general function defined in (5.2.8) is recently studied by Kumar [56] for $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$ and by Jaimini and Tatwal [43] for $\phi(n)=\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{n!\prod_{j=1}^{q}(\sigma_j)_{\xi_j n}}$

In this chapter we have studied the following multivariable analogue of the general function studied by Kumar [56].

The multivariable general function considered here is defined and represented as follows:

$$\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}(z_1,\ldots,z_r;s,a)=\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}p_i n_i\right)^{s}\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_i n_i}\,z_i^{n_i}}{n_i\,!}\right]$$

…(5.2.9)

Where for all $i=1,2,\ldots,r;\ \mu_i,\alpha_i,s,z\in C;\ \delta_i\in R^{+};\ \min\{\operatorname{Re}(a),\operatorname{Re}(s)\}>0;$ $\max_{1\le i\le r}|z_i|<1,\quad \operatorname{Re}(\alpha_i)>0;\ \operatorname{Re}(p_i)>0.$

The Riemann-Liouville (RL) fractional integral and fractional derivative operators are defined as follows (see [20, Chapter 13], [48, pp.69-70] and [87])

$$\left({}^{RL}I_{\alpha+}^{\mu}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_{\alpha}^{x}(x-t)^{\mu-1}f(t)\,dt \qquad \ldots(5.2.10)$$

$$\left(x>\alpha;\operatorname{Re}(\mu)>0\right)$$

$$\left({}^{RL}I_{\alpha-}^{\mu}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_{x}^{\alpha}(t-x)^{\mu-1}f(t)\,dt \qquad \ldots(5.2.11)$$

$$\left(x<\alpha;\operatorname{Re}(\mu)>0\right)$$

and

$$\left({}^{RL}D_{\alpha\pm}^{\mu}f\right)(x)=\left(\pm\frac{d}{dx}\right)^{n}\left(I_{\alpha\pm}^{n-\mu}f\right)(x) \qquad \ldots(5.2.12)$$

$$\left(\operatorname{Re}(\mu)>0\ ;n=\left[\operatorname{Re}(\mu)\right]+1\right)$$

where the function $f$ is locally integrable, $\mathrm{Re}(\mu)$ denotes the real part of the complex number $\mu \in C$ and $[\mathrm{Re}(\mu)]$ denotes the greatest integer in $\mathrm{Re}(\mu)$ and $\Gamma z$ is the classical Gamma function of argument $z$, defined by

$$\Gamma z = \begin{cases} \int_0^\infty e^{-t} t^{z-1} & (\mathrm{Re}(z) > 0) \\ \dfrac{\Gamma(z+n)}{\prod_{j=0}^{n-1}(z+j)} & (z \in C/Z_0^- ; n \in N) \end{cases} \quad \ldots(5.2.13)$$

The Laplace transform of a function $f(t)$ is defined as follows

$$L\{f(t);s\} = \int_0^\infty e^{-st} f(t) = F(s) \qquad (\mathrm{Re}(s) > 0) \quad \ldots(5.2.14)$$

The Laplace transform of fractional derivative $\left({}^{RL}D_{0+}^{\mu} f\right)(t)$ is given as follows [76, p.105, eq.(2.2.48)]:

$$L\left\{\left({}^{RL}D_{0+}^{\mu} f\right)(t);s\right\} = s^{\mu} F(s) - \sum_{k=0}^{\infty} s^k \left({}^{RL}D_{0+}^{\mu-k-1} f\right)(0+) \quad \ldots(5.2.15)$$

$$(\mathrm{n}-1 < \mathrm{Re}(\mu) < n \ ; \ n \in N)$$

or equivalently [48, p.84, eq. (2.2.37)]:

$$L\left\{\left({}^{RL}D_{0+}^{\mu} f\right)(t);s\right\} = s^{\mu} F(s) - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{dt^{n-k-1}} \left\{\left({}^{RL}I_{0+}^{n-\mu} f\right)(t)\right\}\Big|_{t=0} \quad \ldots(5.2.16)$$

$$(\mathrm{n}-1 < \mathrm{Re}(\mu) < n \ ; \ n \in N)$$

where $\left({}^{RL}D_{0+}^{\mu-k-1} f\right)(0+) = \left({}^{RL}D_{0+}^{\mu-k-1} f\right)(t)\Big|_{t=0}$ and

$$\frac{d^k}{dt^k}\left\{\left({}^{RL}I_{0+}^{n-\mu} f\right)(t)\right\}\Big|_{t=0} = \frac{d^k}{dt^k}\left\{\left({}^{RL}I_{0+}^{n-\mu} f\right)(0+)\right\}$$

$$(k \in \{0,1,\ldots,(n-1)\})$$

For the ordinary derivatives $f^{(n)}(t)$ of order $n \in N_0$, it is known that

$$L\left\{f^{(n)}(t);s\right\} = s^{\mu} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+) \quad \ldots(5.2.17)$$

Liouville [62] and Caputo [6] defined a fractional derivative of order $\mu > 0$ of a function $f(t)$ as follows.

$$\frac{d^\mu}{dt^\mu}\{f(x)\} = \left({}^{LC}D_{0+}^{\mu} f\right)(x) = \begin{cases} f^{(n)}(x) & , \quad \mu = n \in N_0 \\ \dfrac{1}{\Gamma(n-\mu)} \displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\mu-n+1}} dt, & \left(n-1 < \mathrm{Re}(\mu) < n; n \in N\right) \end{cases}$$

…(5.2.18)

where $n = \begin{cases} \left[\mathrm{Re}(\mu)\right] + 1 & (\mu \neq N_0) \\ \mu & (\mu \in N_0) \end{cases}$ …(5.2.19)

The Laplace transform formula for the Liouville-Caputo fractional derivative $\left({}^{LC}D_{0+}^{\mu} f\right)(t)$ is as follows ([76, p.80, eq. (2.140)] and [48, p.98, eq. (2.4.62)])

$$L\left\{\left({}^{LC}D_{0+}^{\mu} f\right)(t); s\right\} = s^\mu F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0+) \quad \text{…(5.2.20)}$$

$\left(\mathrm{n}-1 < \mathrm{Re}(\mu) \le n \ ; \ n \in N\right)$

The Laplace transform of the function defined in (1.9) is easily obtained

$$L\left\{t^{\gamma-1} \in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)} \left(\mathrm{z}_1 t^{\nu_1}, \ldots, \mathrm{z}_r t^{\nu_r}; \mathrm{k}, a\right); s\right\} = \frac{1}{s^\gamma} \sum_{n_1,\ldots,\mathrm{n}_r=0}^{\infty} \frac{\Gamma\left(\sum_{i=1}^{r} \nu_i n_i + \gamma\right)}{\left(a + \sum_{i=1}^{r} p_i n_i\right)^k \Gamma\left(\beta + \sum_{i=1}^{r} \alpha_i n_i\right)}$$

$$\times \prod_{i=1}^{r} \left[\frac{(\mu_i)_{\delta_i n_i}}{n_i!} \left(\frac{z_i}{s^{\nu_i}}\right)^{n_i}\right] \quad \text{…(5.2.21)}$$

($\forall\ i = 1, 2, \ldots, r;\ \mathrm{Re}(k) > 0; \mathrm{Re}(\gamma) > 0; \mathrm{Re}(\nu_i) > 0; \mathrm{Re}(\alpha_i) > 0$)

provided that each number of (5.2.21) exists.

For $\gamma = \beta$ and $\nu_i = \alpha_i \ (i = 1, 2, \ldots, r)$ the formula (5.2.21) reduces to the following form:

$$L\left\{t^{\beta-1}\phi_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1 t^{\alpha_1},...,z_r t^{\alpha_r};k,a\right);s\right\}=\frac{1}{s^{\gamma}}\sum_{n_1,..,n_r=0}^{\infty}\frac{1}{\left(a+\sum_{i=1}^{r}p_i n_i\right)^k}\prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_i n_i}}{n_i\,!}\left(\frac{z_i}{s^{\alpha_i}}\right)^{n_i}\right]$$

…(5.2.22)

( $\forall\ i=1,2,...,r;\ \mathrm{Re}(k)>0;\mathrm{Re}(\alpha_i)>0;\mathrm{Re}(\beta)>0$ )

If we apply the limit formula in eq. (5.2.9) then we have

$$E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r\right)=\lim_{s\to 0}\left\{\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;s,a\right)\right\}$$ …(5.2.23)

The multivariable Mittag-Leffler function is due to Gupta [30] (see also [88]) defined as follows

$$E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r\right)=\sum_{n_1,..,n_r=0}^{\infty}\frac{1}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_i n_i}\,z_i^{n_i}}{n_i\,!}\right]$$ …(5.2.24)

( $\forall\ i=1,2,...,r;\beta,\mu_i,\alpha_i,z\in C,\ \mathrm{Re}(\alpha_i)>0,\mathrm{Re}(\delta_i)>0$ )

In view of (5.2.23), the formula (5.2.21) provides the following Laplace transform for multivariable Mittag-Leffler function defined in (5.2.24).

$$L\left\{t^{\gamma-1}E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1 t^{v_1},...,z_r t^{v_n}\right);s\right\}=\frac{1}{s^{\gamma}}\sum_{n_1,..,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left\{\frac{(\mu_i)_{\delta_i n_i}}{n_i\,!}\left(\frac{z_i}{s^{v_i}}\right)^{n_i}\right\}$$

…(5.2.25)

( $\mathrm{Re}(s)>0;\mathrm{Re}(\gamma)>0;\mathrm{Re}(v_i)>0;\mathrm{Re}(\alpha_i)>0;\ \forall\ i=1,2,...,r$ )

For $\gamma=\beta$ and $v_i=\alpha_i\ (i=1,2,...,r)$ the formula (5.2.25) reduces to the known result due to the Gupta [30, p.233, eq.(4.7.6)] in the following form

$$L\left\{t^{\beta-1}E_{(\alpha_r),\beta}^{(\mu_r),(\delta_r)}\left(z_1 t^{\alpha_1},...,z_r t^{\alpha_r}\right);s\right\}=\frac{1}{s^{\beta}}\sum_{n_1,..,n_r=0}^{\infty}\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}}{(n_i)!}\left(\frac{z_i}{s^{\alpha_i}}\right)^{n_i}$$ …(5.2.26)

The Laplace transform formula for Hurwitz-Lerch zeta function defined in (5.2.5) is given by

$$L\left\{t^{\gamma-1}\phi^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}\left(z_1 t^{v_1},...,z_r t^{v_r};u,a\right);s\right\}=\frac{\Gamma\gamma}{s^{\gamma}}\Phi^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}\left(z_1 s^{-v_1},...,z_r s^{-v_r};u,a\right)$$

…(5.2.27)

where

$$\Phi^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}\left(z_1,...,z_r;u,a\right)=\sum_{n_1,...,n_r=0}^{\infty}\frac{(\gamma)_{\sum_{i=1}^{r}v_i}}{\left(a+\sum_{i=1}^{r}\theta_i n_i\right)^u}\prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}}\frac{z_i^{n_i}}{n_i!}\right\} \quad \text{…(5.2.28)}$$

$(\forall\ i=1,2,...,r;\ \operatorname{Re}(u)>0;\operatorname{Re}(\gamma)>0;\operatorname{Re}(v_i)>0)$

the result (5.2.27) holds good for the conditions given with (5.2.5) for the similar notations used their in (5.2.5).

Formally, the kinetic equation can be described by the following equation

$$\frac{dN}{dt}=-d\left(N_t\right)+p\left(N_t\right) \quad \text{…(5.2.29)}$$

where $d$ is the destruction rate and $p$ is the production rate of $N(t)$, also

$$N_t\left(t^*\right)=N\left(t-t^*\right), \qquad \left(t^*>0\right)$$

Haubold and Mathai [34] studied the elementary kinetic equation (5.2.29) is the following form

$$\frac{dN_i}{dt}=-c_i N_i\left(t\right) \quad \text{…(5.2.30)}$$

with the initial condition that $N_i(t=0)=N_0$ is the number of density of species $i$ at time $t=0$ and constant $c_i>0$, is known as the standard kinetic equation.

An alternative form of equation (5.2.30) upon integration is given by

$$N(t)-N_0=c\,{}_0D_t^{-1}\left\{N(t)\right\} \quad \text{…(5.2.31)}$$

where ${}_0D_t^{-1}$ is the standard integral operator

The fractional order generalization of the equation (5.2.31) is recently being studied in the following form (see [34])

$$N(t)-N_0=c^v\left({}^{RL}I_{0+}^{v}N\right)(t) \quad \text{…(5.2.32)}$$

where the Riemman-Liouville integral operator ${}^{RL}I_{0+}^{v}$ is of order $v$ defined in (5.2.10) (see also [48])

## SECTION-A

**5.3** In this section, we have established three theorems. The solution of the general hybrid-type fractional order kinetic equations involving the multivariable general function, multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta function for the Liouville-Caputo fractional derivative operator ${}^{LC}D_{0+}^{\sigma}$ are obtained here. These are solved by using the Laplace transform method.

The kinetic equations considered in this section are very general and provide the solutions of certain known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving the Hurwitz-Lerch Zeta function of one and several variables.

In this section-A, we investigate the solution of following general hybrid-type fractional order kinetic equations involving the multivariable general functions defined in (5.2.5), (5.2.9) and (5.2.24) for the Liouville-Caputo fractional derivative operator ${}^{LC}D_{0+}^{\sigma}$ defined in (5.2.18).

## 5.4 MAIN KINETIC EQUATIONS WITH LIOUVILLE-CAPUTO DERIVATIVE

The kinetic equations and their solutions are given in the following Theorem-1, 2 and 3 as below.

**Theorem-1**

Under the parametric constraints $c,\gamma,\nu,\rho \in R^{+}$ and $0<\sigma<1$, let the general function $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,\mathrm{z}_r;s,a\right)$ defined in (5.2.9), exists then the solution of the following generalized fractional order kinetic equation

$$N\left(t\right)-N_0 t^{\gamma-1}\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(\mathrm{z}_1 t^{\nu_1},...,\mathrm{z}_r\, t^{\nu_r};k,a\right)=-c^{\rho}\left({}^{LC}D_{0+}^{\sigma}N\right)\left(t\right) \qquad \ldots(5.4.1)$$

with initial condition $N\left(0+\right)=N\left(t\right)\Big|_{t=0}=\chi_0$ …(5.4.2)

is given by

$$N(t) = N_0 t^{\gamma-1} \sum_{q=0}^{\infty} (-1)^q \left(\frac{t^\sigma}{c^\rho}\right)^{q+1} \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\Gamma\left(\sum_{i=1}^{r} v_i n_i + \gamma\right)}{\left(a + \sum_{i=1}^{r} p_i n_i\right)^k \Gamma\left(\beta + \sum_{i=1}^{r} \alpha_i n_i\right)}$$

$$\frac{1}{\Gamma\left(\sum_{i=1}^{r} v_i n_i + \sigma(q+1) + \gamma\right)} \prod_{i=1}^{r} \left[\frac{(\mu_i)_{\delta_i n_i} \left(z_i t^{v_i}\right)^{n_i}}{n_i!}\right] + \chi_0 \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma \sigma q + 1} \left(\frac{t^\sigma}{c^\rho}\right)^q$$

$$(t > 0) \qquad \ldots(5.4.3)$$

provided that the right hand side of the solution (5.4.3) exists.

**Proof of Theorem-1**

On applying the Laplace transform of both side of kinetic equation (5.4.1) and using the formulae (5.2.20), (5.2.21) and (5.4.2) we have

$$N(s) - \frac{N_0}{s^\gamma} \left[\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\Gamma\left(\sum_{i=1}^{r} v_i n_i + \gamma\right)}{\left(a + \sum_{i=1}^{r} p_i n_i\right)^k \Gamma\left(\beta + \sum_{i=1}^{r} \alpha_i n_i\right)} \prod_{i=1}^{r} \left\{\frac{(\mu_i)_{\delta_i n_i}}{n_i!} \left(\frac{z_i}{s^{v_i}}\right)^{n_i}\right\}\right]$$

$$= -c^\rho s^\sigma N(s) + c^\sigma s^{\sigma-1} \chi_0$$

which leads the following result

$$N(s) = \frac{N_0}{1 + c^\rho s^\sigma} \left[\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\Gamma\left(\sum_{i=1}^{r} v_i n_i + \gamma\right)}{\left(a + \sum_{i=1}^{r} p_i n_i\right)^k \Gamma\left(\beta + \sum_{i=1}^{r} \alpha_i n_i\right)} \left\{\prod_{i=1}^{r} \frac{(\mu_i)_{\delta_i n_i} z_i^{n_i}}{n_i!}\right\} \frac{1}{s^{\left(\sum_{i=1}^{r} v_i n_i + \gamma\right)}}\right]$$

$$+ \frac{c^\rho \chi_0}{1 + c^\rho s^\sigma} s^{\sigma-1}$$

on applying the series expansion given as follows

$$\frac{1}{1 + c^\rho s^\sigma} = \sum_{q=0}^{\infty} \frac{(-1)^q}{\left(c^\rho s^\sigma\right)^{q+1}} \qquad \left(\left|c^\rho s^\sigma\right| > 1\right) \qquad \ldots(5.4.4)$$

then above expression can be written in the following form

$$N\left(s\right)=N_0\sum_{q=0}^{\infty}\frac{\left(-1\right)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\dots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}p_in_i\right)^k\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\right.$$

$$\left.\left\{\prod_{i=1}^{r}\frac{\left(\mu_i\right)_{\delta_in_i}z_i^{n_i}}{n_i!}\right\}\frac{1}{s^{\left(\sum_{i=0}^{r}v_in_i+\gamma+\sigma(q+1)\right)}}\right]+\chi_0\sum_{q=0}^{\infty}\frac{\left(-1\right)^q}{c^{\rho q}s^{\sigma q+1}}$$

now applying the inversion of Laplace transform by using the well known identity given as follows

$$L^{-1}\left\{\frac{1}{s^{\lambda+1}};t\right\}=\frac{t^{\lambda}}{\Gamma\left(\lambda+1\right)} \qquad \dots(5.4.5)$$

$$\left(\operatorname{Re}\left(\lambda\right)>-1;\operatorname{Re}\left(s\right)>0\right)$$

We have

$$N\left(t\right)=N_0\sum_{q=0}^{\infty}\frac{\left(-1\right)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\dots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}p_in_i\right)^k\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\right.$$

$$\left.\left\{\prod_{i=1}^{r}\frac{\left(\mu_i\right)_{\delta_in_i}z_i^{n_i}}{n_i!}\right\}\frac{t^{\left(\sum_{i=0}^{r}v_in_i+\gamma+\sigma(q+1)-1\right)}}{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma+\sigma(q+1)\right)}\right]+\chi_0\sum_{q=0}^{\infty}\frac{\left(-1\right)^q}{c^{\rho q}}\frac{t^{\sigma q}}{\Gamma\sigma q+1}$$

On simplifying above expression, we obtain the solution (5.4.3) asserted by Theorem-1

**Theorem-2**

Let $c,\gamma,v,\rho\in R^{+}$ and $0<\sigma<1$. Also let the multivariable Mittag-Leffler function $E_{(\alpha_r),\beta}^{(\mu_r),(\delta_r)}\left(z_1,\dots,z_r\right)$ defined in (5.2.24) exists. If $\chi_0$ is given by (5.4.2), then the solution of the following generalized fractional order kinetic equation

$$N\left(t\right)-N_0t^{\gamma-1}E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}\left(z_1t^{v_1},\dots,z_r\,t^{v_r}\right)=-c^{\rho}\left({}^{LC}D_{0+}^{\sigma}N\right)\left(t\right) \qquad \dots(5.4.6)$$

is given by

$$N(t)=N_0 t^{\gamma-1}\sum_{q=0}^{\infty}(-1)^q\left(\frac{t^\sigma}{c^\rho}\right)^{q+1}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma+\sigma(q+1)\right)}$$

$$\times\prod_{i=1}^{r}\left\{\frac{(\mu_i)_{\delta_i n_i}\left(z_i t^{v_i}\right)^{n_i}}{n_i!}\right\}+\chi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{\Gamma\sigma q+1}\left(\frac{t^\sigma}{c^\rho}\right)^q \qquad (t>0) \qquad \ldots(5.4.7)$$

provided that the right hand side of the solution (5.4.7) exists

**Proof of Theorem-2**

Applying the Laplace transform of both side of kinetic equation (5.4.6) and using the equation (5.2.25), (5.2.20) and (5.4.2), we have

$$N(s)-\frac{N_0}{s^\gamma}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\prod_{i=1}^{r}\left\{\frac{(\mu_i)_{\delta_i n_i}}{n_i!}\left(\frac{z_i}{s^{v_i}}\right)^{n_i}\right\}\right]$$

$$=-c^\rho s^\sigma N(s)+c^\sigma s^{\sigma-1}\chi_0$$

Which leads the following result

$$N(s)=\frac{N_0}{1+c^\rho s^\sigma}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}z_i^{n_i}}{n_i!}\right\}\frac{1}{s^{\sum_{i=1}^{r}v_i n_i+\gamma}}\right]+\frac{c^\rho\chi_0}{1+c^\rho s^\sigma}s^{\sigma-1}$$

On applying the series expansion given in equation (5.4.4) above expression can be written as follows

$$N(s)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)^k}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}z_i^{n_i}}{n_i!}\right\}\frac{1}{s^{\left(\sum_{i=1}^{r}v_i n_i+\gamma+\sigma(q+1)\right)}}\right]$$

$$+\chi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho q}s^{\sigma q+1}}$$

On applying the inverse Laplace transform (5.4.5), we have

$$N(t)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}\, z_i^{n_i}}{n_i\,!}\right\}\frac{t^{\left(\sum_{i=0}^{r}v_i n_i+\gamma+\sigma(q+1)-1\right)}}{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma+\sigma(q+1)\right)}\right]$$

$$+\chi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho q}}\frac{t^{\sigma q}}{\Gamma\sigma q+1}$$

On simplifying above expression, we obtain the solution (5.4.7) asserted by Theorem-2.

**Theorem-3**

For $c,\gamma,v,\rho\in R^{+}$ and $0<l<1$, let the extended Hurwitz-Lerch Zeta function of multivariable defined in (5.2.5) exists. If $\chi_0$ is given by (5.4.2), then the solution of the following generalized fractional order kinetic equation

$$N(t)-N_0 t^{\gamma-1}\phi_{\theta_i;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1 t^{v_1},\ldots,\mathrm{z}_r\, t^{v_r};u,a\right)=-c^{\rho}\left({}^{LC}D_{0+}^{l}N\right)(t) \qquad \ldots(5.4.8)$$

is given by

$$N(t)=N_0 t^{\gamma-1}\sum_{q=0}^{\infty}(-1)^q\left(\frac{t^l}{c^{\rho}}\right)^{q+1}\frac{\Gamma\gamma}{\Gamma(l(q+1)+\gamma)}\Phi_{\theta_i;l(q+1)+\gamma;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1 t^{v_1},\ldots,z_r t^{v_r};u,a\right)$$

$$+\chi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{\Gamma lq+1}\left(\frac{t^l}{c^{\rho}}\right)^q \qquad (t>0) \qquad \ldots(5.4.9)$$

provided that the right hand side of the solution (5.4.9) exists and the function $\Phi_{\theta_i;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,\ldots,z_r;u,a\right)$ is defined in (5.2.28).

**Proof of Theorem-3**

On applying the Laplace transform of both side of kinetic equation (5.4.8) and using the formulae (5.2.20), (5.2.27) and (5.4.2), we have

$$N(s)-\frac{N_0}{s^{\gamma}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_in_i\right)^u}\prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\left(\frac{z_i}{s^{v_i}}\right)^{n_i}\right\}\right]$$

$$= -c^{\rho}s^{l}N(s)+c^{\rho}s^{l-1}\chi_0$$

Which leads the following result

$$N(s)=\frac{N_0}{1+c^{\rho}s^{l}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_in_i\right)^u}\left\{\prod_{i=1}^{r}\left(\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\right)\right\}\frac{1}{s^{\left(\sum_{i=0}^{r}v_in_i+\gamma\right)}}\right]+\frac{c^{\rho}\chi_0}{1+c^{\rho}s^{l}}s^{l-1}$$

Now in view of series expansion (5.4.4) it can be written as follows

$$N(s)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_in_i\right)^u}\left\{\prod_{i=1}^{r}\left(\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\right)\right\}\frac{1}{s^{\sum_{i=1}^{r}v_in_i+\gamma+l(q+1)}}\right]$$

$$+\chi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho q}s^{lq+1}}$$

on applying the inverse Laplace transform on both side in view of (5.4.5), we have

$$N(t)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_in_i\right)^u}\left\{\prod_{i=1}^{r}\left(\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\right)\right\}\right.$$

$$\left.\frac{t^{\left(\sum_{i=1}^{r}v_in_i+\gamma+l(q+1)-1\right)}}{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma+l(q+1)\right)}\right]+\chi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho q}}\frac{t^{lq}}{\Gamma lq+1}$$

On simplifying above expression, we obtain the solution (5.4.9) asserted by Theorem-3.

## 5.5 APPLICATIONS

The kinetic equations in Theorem-1, 2 and 3 are providing solutions of a significantly large number of known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving Hurwitz-Lerch Zeta function of one and several variables. We mention some of them as below.

(i) At $r=1$ and $p_1=1$, the Theorem-1 reduce to the known result due to Srivastava [94, p. 2663, Corollary 4.6 at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] for the function $\in_{\alpha,\beta}\left(\phi;z,s,a\right)$ considered therein [94].

(ii) At $r=1$, the Theorem-2 reduce to the known result due to Srivastava [94, p. 2663, Corollary 4.7 at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] for Mittag-Leffler function.

(iii) At $r=1$ and for $\theta_1=1, p_1=p$ and $q_1=q$, the Theorem-3 reduce to the known result due to Srivastava [94, p. 2664, Corollary 4.8].

(iv) At $r=1, p_1=1$ and if $\chi_0$ is replaced by $\sigma q\ \chi_0$, the Theorem-1 reduce to the known results due to Srivastava [103, p. 56, eq.(3.10) at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] for the function $\in_{\alpha,\beta}\left(\phi;z,s,a\right)$ considered therein [103].

(v) At $r=1$ and if $\chi_0$ is replaced by $\sigma q\ \chi_0$, the Theorem-2 reduce to the known result due to Srivastava [103, p. 57, eq.(3.19) at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] for Mittag-Leffler function.

(vi) At $r=1, \theta_1=1, p_1=p, q_1=q$ and if $\chi_0$ is replaced by $\sigma q\ \chi_0$, the Theorem-3 reduce to the known result due to Srivastava [103, p. 58, eq.(3.21)].

## SECTION-B

**5.6** Recently Srivastava [94] studied some general families of hybrid-type fractional order kinetic equations for the Hilfer type fractional derivative operator involving a general class of functions using Laplace transform method. Following the above cited work of Srivastava [94], three theorems on solution of general families of hybrid-type fractional order kinetic equations for the Hilfer type fractional derivative operator ${}^{H}D_{0+}^{\sigma,\omega}$ involving multivariable general function associated with multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch zeta function defined in (5.2.5), (5.2.9) and (5.2.24) are obtained here in this section.

As applications of the theorems established in this section, some known and new results on fractional order kinetic equations and their solutions involving multivariable general functions and associated functions for Riemann-Liouville fractional derivative ${}^{RL}D_{0+}^{\sigma}$ and Liouville-Caputo fractional derivative operator ${}^{LC}D_{0+}^{\sigma}$ are established here.

## 5.7 DEFINITIONS AND RESULT REQUIRED

The Two-parameter family of fractional derivatives of order is $(0<\mu<1)$ and type $v(0\le v\le 1)$ was introduced and studied recently by Hilfer in the following manner (see [36,37,38,39,112]):

$$\left({}^{H}D_{a\pm}^{\mu,\mathrm{v}}f\right)(x)=\pm\left({}^{RL}I_{a\pm}^{v(1-\mu)}\frac{d}{dx}\left({}^{RL}I_{a\pm}^{(1-v)(1-\mu)}f\right)\right)(x) \qquad \ldots(5.7.1)$$

Here, the right sided and left sided Hilfer fractional derivative of order $\mu$ and type $v$ with respect to $x$ are defined, in terms of the Riemann-Liouville fractional integral operator defined in (5.2.10) and (5.2.11).

The Laplace transform of Hilfer fractional derivative $\left({}^{H}D_{0+}^{\mu,\mathrm{v}}f\right)(t)$ is given as follows [94, p.2651, eq.(1.15)]:

$$L\left\{\left({}^{H}D_{0+}^{\mu,\mathrm{v}}f\right)(t);s\right\}=s^{\mu}L\left\{f(t);s\right\}-s^{v(\mu-1)}\left({}^{RL}I_{0+}^{(1-v)(1-\mu)}f\right)(0+) \qquad \ldots(5.7.2)$$

where $\left({}^{RL}I_{0+}^{(1-v)(1-\mu)}f\right)(0+)$ is the Riemann-Liouville integral of order $(1-v)(1-\mu)$.

## 5.8 MAIN KINETIC EQUATIONS WITH HILFER DERIVATIVE

The three kinetic equations and their solution are given in the Theorem-1, 2 and 3. These kinetic equations are solved using the Laplace transform method.

**Theorem-1**

Let each of the parametric constraints $\gamma,\rho\ ,c,v_i \in R^{+}\ \left(i=1...r\right)$, $0<\sigma<1$ and $0\le\omega\le1$ hold true. Suppose also that the multivariable general function $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,z_r;k,a\right)$ defined by equation (5.2.9) exists. If we consider

$$\chi_0\left(\sigma,\omega\right)=\left({}^{RL}I_{0+}^{(1-\omega)(1-\sigma)}N\right)\left(0+\right) \qquad \ldots(5.8.1)$$

then the solution of the following general hybrid-type family of the fractional-order kinetic equations.

$$N\left(t\right)-N_0 t^{\gamma-1}\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1 t^{v_1},...,z_r\, t^{v_r};\mathrm{k},a\right)=-c^{\rho}\left({}^{H}D_{0+}^{\sigma,\omega}N\right)\left(t\right) \qquad \ldots(5.8.2)$$

is given by

$$N\left(t\right)=N_0 t^{\gamma-1}\sum_{q=0}^{\infty}\left(-1\right)^q\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q+1}\sum_{n_1,...,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\left(a+\sum_{i=1}^{r}p_i n_i\right)^k}$$

$$\frac{1}{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma+\sigma\left(q+1\right)\right)}\prod_{i=1}^{r}\left[\frac{\left(\mu_i\right)_{\delta_i n_i}\left(z_i t^{v_i}\right)^{n_i}}{n_i\,!}\right]$$

$$+\chi_o\left(\sigma,\omega\right)t^{\sigma+\omega(1-\sigma)-1}\sum_{q=0}^{\infty}\frac{\left(-1\right)^q}{\Gamma\left[\sigma\left(q+1\right)+\omega\left(1-\sigma\right)\right]}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^q \qquad \left(t>0\right)$$

$$\ldots(5.8.3)$$

provided that the right hand side of the solution (5.8.3) exists.

**Proof of Theorem-1**

On applying the Laplace transform of both side of (5.8.2) then using (5.2.21), (5.7.2) and (5.8.1) we have

$$N(s)-\frac{N_0}{s^{\gamma}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\left(a+\sum_{i=1}^{r}p_i n_i\right)^k}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}}{n_i!}\left(\frac{z_i}{s^{v_i}}\right)^{n_i}\right\}\right]$$

$$=-c^{\rho}\left[s^{\sigma}N(s)-s^{\omega(\sigma-1)}\chi_o(\sigma,\omega)\right]$$

which leads the following result

$$N(s)=\frac{N_0}{1+c^{\rho}s^{\sigma}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\left(a+\sum_{i=1}^{r}p_i n_i\right)^k}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}z_i^{n_i}}{n_i!}\right\}\frac{1}{s^{\left(\sum_{i=0}^{r}v_i n_i+\gamma\right)}}\right]$$

$$+\frac{c^{\rho}\chi_0(\sigma,\omega)}{1+c^{\rho}s^{\sigma}}s^{\omega(\sigma-1)}$$

on applying the series expansion (5.4.4), the above expression can be written in the following form

$$N(s)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\left(a+\sum_{i=1}^{r}p_i n_i\right)^k}\right.$$

$$\left.\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}z_i^{n_i}}{n_i!}\right\}\frac{1}{s^{\left(\sum_{i=1}^{r}v_i n_i+\gamma+\sigma(q+1)\right)}}\right]+\chi_0(\sigma,\omega)\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho q}s^{\sigma(q+1)+\omega(1-\sigma)}}$$

now applying the inversion of Laplace transform by using (5.4.5), we have

$$N(t)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\left(a+\sum_{i=1}^{r}p_i n_i\right)^k}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_i n_i}z_i^{n_i}}{n_i!}\right\}\right.$$

$$\left.\frac{t^{\sum_{i=1}^{r}v_i n_i+\gamma+\sigma(q+1)-1}}{\Gamma\left[\sum_{i=1}^{r}v_i n_i+\gamma+\sigma(q+1)\right]}\right]+\chi_0(\sigma,\omega)\sum_{q=0}^{\infty}\frac{(-1)^q}{C^{\rho q}}\frac{t^{\sigma(q+1)+\omega(1-\sigma)-1}}{\Gamma\left[\sigma(q+1)+\omega(1-\sigma)\right]}$$

on simplifying above expression, we obtain the solution (5.8.3) asserted by Theorem-1.

**Theorem-2**

Let each of the parametric constraints $c,\gamma,\nu,\rho\in R^{+}$, $0<\sigma<1$, and $0\le\omega\le 1$ hold true. Suppose also that the multivariable Mittag-Leffler function $E_{(\alpha_r),\beta}^{(\mu_r),(\delta_r)}\left(z_1,...,z_r\right)$ defined in (5.2.24) exists. If $\chi_0\left(\sigma,\omega\right)$ is defined by the equation (5.8.1), then the solution of the following general hybrid type family of fractional order kinetic equations

$$N\left(t\right)-N_0 t^{\gamma-1}E_{(\alpha_r),\beta}^{(\mu_r),(\delta_r)}\left(z_1 t^{\nu_1},...,z_r t^{\nu_r}\right)=-c^{\rho}\left({}^{H}D_{0+}^{\sigma,\omega}N\right)\left(t\right) \qquad \text{…(5.8.4)}$$

is given by

$$N\left(t\right)=N_0 t^{\gamma-1}\sum_{q=0}^{\infty}\left(-1\right)^{q}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q+1}\sum_{n_1,...,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}\nu_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\Gamma\left(\sum_{i=1}^{r}\nu_i n_i+\gamma+\sigma\left(q+1\right)\right)}$$

$$\prod_{i=1}^{r}\left[\frac{\left(\mu_i\right)_{\delta_i n_i}\left(z_i t^{\nu_i}\right)^{n_i}}{n_i\,!}\right]+\chi_o\left(\sigma,\omega\right)t^{\sigma+\omega(1-\sigma)-1}\sum_{q=0}^{\infty}\frac{\left(-1\right)^{q}}{\Gamma\left[\sigma\left(q+1\right)+\omega\left(1-\sigma\right)\right]}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q}$$

$$\left(t>0\right) \qquad \text{…(5.8.5)}$$

provided that the right hand side of the solution (5.8.5) exists.

**Proof of Theorem-2**

On applying the Laplace transform of both side of kinetic equation (5.8.4) and using the equation (5.2.25), (5.7.2) and (5.8.1), we have

$$N\left(s\right)-\frac{N_0}{s^{\gamma}}\left[\sum_{n_1,...,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}\nu_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\left\{\prod_{i=1}^{r}\frac{\left(\mu_i\right)_{\delta_i n_i}}{n_i\,!}\left(\frac{z_i}{s^{\nu_i}}\right)^{n_i}\right\}\right]$$

$$=-c^{\rho}\left[s^{\sigma}N\left(s\right)-s^{\omega(\sigma-1)}\chi_0\left(\sigma,\omega\right)\right]$$

which leads to the following result

$$N\left(s\right)=\frac{N_0}{1+c^{\rho}s^{\sigma}}\left[\sum_{n_1,...,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}\nu_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)}\left\{\prod_{i=1}^{r}\frac{\left(\mu_i\right)_{\delta_i n_i}z_i^{n_i}}{\left(n_i\right)!}\right\}\frac{1}{s^{\left(\sum_{i=1}^{r}\nu_i n_i+\gamma\right)}}\right]+\frac{c^{\rho}\chi_0\left(\sigma,\omega\right)}{1+c^{\rho}s^{\sigma}}s^{\omega(\sigma-1)}$$

on applying the series expansion (5.4.4), the above expression can be written in the following form

$$N(s)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_in_i}\,z_i^{n_i}}{(n_i)!}\right\}\frac{1}{s^{\left(\sum_{i=1}^{r}v_in_i+\gamma+\sigma(q+1)\right)}}\right]$$

$$+\chi_0(\sigma,\omega)\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho q}s^{\sigma(q+1)+\omega(1-\sigma)}}$$

now on applying the inverse Laplace transform in view of (5.4.5) we have

$$N(t)=N_0\sum_{q=0}^{\infty}\frac{(-1)^q}{c^{\rho(q+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)^k}\left\{\prod_{i=1}^{r}\frac{(\mu_i)_{\delta_in_i}\,z_i^{n_i}}{n_i\,!}\right\}\frac{t^{\sum_{i=1}^{r}v_in_i+\gamma+\sigma(q+1)-1}}{\Gamma\left[\sum_{i=1}^{r}v_in_i+\gamma+\sigma(q+1)\right]}\right]$$

$$+\chi_0(\sigma,\omega)\sum_{q=0}^{\infty}\frac{(-1)^q\,t^{[\sigma(q+1)+\omega(1-\sigma)-1]}}{c^{\rho q}\,\Gamma[\sigma(q+1)+\omega(1-\sigma)]}$$

On simplifying the above expression, we obtain the solution (5.8.5) asserted by Theorem-2.

**Theorem-3**

Let each of the parametric constraints $c,\gamma,v,\rho\in R^+$, $0<l<1$, and $0\le\omega\le1$ hold true. Suppose that the extended multivariable Hurwitz-Lerch Zeta function which is defined in (5.2.5) exists. If $\chi_0(l,\omega)$ is defined by the equation (5.8.1), then the solution of the following general hybrid type family of fractional order kinetic equations

$$N(t)-N_0t^{\gamma-1}\phi^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right),\left(\xi_k^{(i)}\right)}\left(z_1t^{v_1},\ldots,z_r\,t^{v_r};u,a\right)=-c^{\rho}\left({}^{H}D_{0+}^{l,\omega}N\right)(t) \qquad \ldots(5.8.6)$$

is given by

$$N(t)=N_0t^{\gamma-1}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t^l}{c^{\rho}}\right)^{m+1}\frac{\Gamma\gamma}{\Gamma(l(m+1)+\gamma)}\Phi^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;(l(m+1)+\gamma);v_i,\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}\left(z_1t^{v_1},\ldots,z_rt^{v_r};u,a\right)$$

$$+\chi_0(t,\omega)t^{[l+\omega(1-l)-1]}\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma[l(m+1)+\omega(1-l)]}\left(\frac{t^l}{c^{\rho}}\right)^m \qquad (t>0) \qquad \ldots(5.8.7)$$

provided that the right hand side of the solution of the equation (5.8.7) exists and the function $\Phi^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}(z_1,\ldots,z_r;u,a)$ is defined in (5.2.28).

**Proof of Theorem-3**

On applying the Laplace transform of both side of kinetic equation (5.8.6) and then using (5.2.27), (5.7.2) and (5.8.1) we have

$$N(s)-\frac{N_0}{s^{\gamma}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_i n_i\right)^u}\prod_{i=1}^{r}\left\{\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\left(\frac{z_i}{s^{v_i}}\right)^{n_i}\right\}\right]$$

$$=-c^{\rho}\left[s^l N(s)-s^{\omega(l-1)}\chi_o(l,\omega)\right]$$

which leads the following result

$$N(s)=\frac{N_0}{1+c^{\rho}s^l}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_i n_i\right)^u}\left\{\prod_{i=1}^{r}\left(\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\right)\right\}\frac{1}{s^{\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}}\right]$$

$$+\frac{c^{\rho}\chi_0(l,\omega)}{1+c^{\rho}s^l}s^{\omega(l-1)}$$

on applying the series expansion (5.4.4), the above expression can be written in the following form

$$N(s)=N_0\sum_{m=0}^{\infty}\frac{(-1)^m}{c^{\rho(m+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_i n_i\right)^u}\left\{\prod_{i=1}^{r}\left(\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\right)\right\}\frac{1}{s^{\sum_{i=1}^{r}v_i n_i+\gamma+l(m+1)}}\right]$$

$$+\chi_0(l,\omega)\sum_{m=0}^{\infty}\frac{(-1)^q}{c^{\rho m}s^{l(m+1)+\omega(1-l)}}$$

on applying the inverse Laplace transform on both side in view of (5.4.5), we have

$$N(t)=N_0\sum_{m=0}^{\infty}\frac{(-1)^m}{c^{\rho(m+1)}}\left[\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\left(a+\sum_{i=1}^{r}\theta_i n_i\right)^u}\left\{\prod_{i=1}^{r}\left(\frac{\prod_{j=1}^{p_i}\left(\mu_j^{(i)}\right)_{\delta_j^{(i)}n_i}z_i^{n_i}}{\prod_{k=1}^{q_i}\left(\sigma_k^{(i)}\right)_{\xi_k^{(i)}n_i}n_i!}\right)\right\}\right.$$

$$\left.\frac{t^{\sum_{i=1}^{r}v_i n_i+\gamma+l(m+1)-1}}{\Gamma\left[\sum_{i=1}^{r}v_i n_i+\gamma+l(m+1)\right]}\right]+\chi_0(l,\omega)\sum_{m=0}^{\infty}\frac{(-1)^q}{c^{\rho m}}\frac{t^{l(m+1)+\omega(1-l)-1}}{\Gamma\left[l(m+1)+\omega(1-l)\right]}$$

On simplifying the above expression, we obtain the solution (5.8.7) asserted by Theorem-3.

## 5.9 APPLICATIONS

The fractional kinetic equations involving multivariable general function for Hilfer fractional order derivative operator studied here are general in nature and provide the wide range of fractional kinetic equations. As applications of theorems established in this section these can be reduced to the results for Liouville-Caputo fractional derivative as well as for Riemann-Liouville fractional integral operator.

Each of the corollaries reduced from our main theorem are significantly general in nature and can provide the solution of many relatively simple fractional order kinetic equations. Certain known and new fractional kinetic equations of our main theorems are illustrated here.

The relationship of Hilfer fractional order derivative with Riemann-Liouville and Liouville-Caputo fractional derivative operator is given in the following manner [94, p.2661, eq.(4.1)]

$$ {}^{RL}D_{o+}^{\sigma} = {}^{H}D_{o+}^{\sigma,0} \qquad \ldots(5.9.1)$$

$$ {}^{LC}D_{o+}^{\sigma} = {}^{H}D_{o+}^{\sigma,1} \qquad \ldots(5.9.2)$$

So our main results can be reduce the solution of the corresponding hybrid-type fractional order kinetic equations involving these simpler fractional derivatives.

At first if we apply (5.9.1), then the Theorem-1, 2 and 3 reduce to the following Corollaries-1, 2 and 3 respectively related to the Riemann-Liouville fractional derivative.

**Corollary-1**

Let each of the parametric constraints $\gamma, \rho\ , c, v_i \in R^{+}\ (i = 1 \ldots r)$, $0 < \sigma < 1$ hold true. Suppose also that the multivariable general function $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}(z_1,\ldots,\mathrm{z}_r;k,a)$ defined by equation (5.2.9) exists. if we set

$$\Upsilon_0(\sigma) = \left({}^{RL}I_{0+}^{(1-\sigma)}N\right)(0+) \qquad \ldots(5.9.3)$$

$(0 < \sigma < 1)$

then the solution of the following general hybrid-type family of the fractional-order kinetic equations

$$N(t)-N_0 t^{\gamma-1}\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1 t^{v_1},\ldots,z_r t^{v_r};k,a\right)=-c^{\rho}\left({}^{RL}D_{0+}^{\sigma}N\right)(t) \qquad \ldots(5.9.4)$$

is given by

$$N(t)=N_0 t^{\gamma-1}\sum_{q=0}^{\infty}(-1)^q\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q+1}\sum_{n_1\ldots n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\left(a+\sum_{i=1}^{r}p_i n_i\right)^k}$$

$$\frac{1}{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\sigma(q+1)+\gamma\right)}\prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_i n_i}\left(z_i t^{v_i}\right)^{n_i}}{n_i!}\right]+\Upsilon_0(\sigma)t^{\sigma-1}\sum_{q=0}^{\infty}\frac{(-1)^q}{\Gamma\left[\sigma(q+1)\right]}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^q$$

$$(t>0) \qquad \ldots(5.9.5)$$

provided that the right hand side of the solution given by the equation (5.9.5) exists.

**Corollary-2**

Let each of the parametric constraints $c,\gamma,v_i,\rho\in R^{+}\,(i=1\ldots r)$, $0<\sigma<1$hold true. Suppose that the multivariable Mittag-Leffler function $E_{(\alpha_r),\beta}^{(\mu_r),(\delta_r)}\left(z_1,\ldots,z_r\right)$defined in (5.2.24) exists. If $\Upsilon_0(\sigma)$is defined by the equation (5.9.3), then the solution of the following general hybrid type family of fractional order kinetic equations

$$N(t)-N_0 t^{\gamma-1}E_{(\alpha_r),\beta}^{(\mu_r),(\delta_r)}\left(z_1 t^{v_1},\ldots,z_r t^{v_r}\right)=-c^{\rho}\left({}^{RL}D_{0+}^{\sigma}N\right)(t) \qquad \ldots(5.9.6)$$

is given by

$$N(t)=N_0 t^{\gamma-1}\sum_{q=0}^{\infty}(-1)^q\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q+1}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_i n_i\right)\Gamma\left(\sum_{i=1}^{r}v_i n_i+\gamma+\sigma(q+1)\right)}$$

$$\times\prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_i n_i}\left(z_i t^{v_i}\right)^{n_i}}{n_i!}\right]+\Upsilon_0(\sigma)t^{\sigma-1}\sum_{q=0}^{\infty}\frac{(-1)^q}{\Gamma\left[\sigma(q+1)\right]}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^q \qquad (t>0)$$

$$\ldots(5.9.7)$$

provided that the right hand side of the solution (5.9.7) exists.

**Corollary-3**

Under the parametric constraints $c,\gamma,v_i,\rho\in R^+\ (i=1...r)$ and $0<l<1$. Suppose that the extended multivariable Hurwitz-Lerch zeta function defined in (5.2.5) exists. If $\Upsilon_0(l)$ is defined by the equation (5.9.3), then the solution of the following generalized hybrid type fractional order kinetic equation

$$N(t)-N_0t^{\gamma-1}\phi^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right),(\xi_k^{(i)})}\left(z_1t^{v_1},...,z_r\,t^{v_r};u,a\right)=-c^{\rho}\left({}^{RL}D^{l}_{0+}N\right)(t) \qquad \text{...(5.9.8)}$$

is given by

$$N(t)=N_0t^{\gamma-1}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t^l}{c^{\rho}}\right)^{m+1}\frac{\Gamma\gamma}{\Gamma\left(l(m+1)+\gamma\right)}\Phi^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(l(m+1)+\gamma\right);v_i,\left(\sigma_k^{(i)}\right);(\xi_k^{(i)})}\left(z_1t^{v_1},...,z_rt^{v_r};u,a\right)$$

$$+\Upsilon_0(l)t^{l-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma\left[l(m+1)\right]}\left(\frac{t^l}{c^{\rho}}\right)^m \qquad (t>0) \qquad \text{...(5.9.9)}$$

provided that the right hand side of the solution of the equation (5.9.9) exists and the function $\Phi^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}_{\theta_i;\left(\sigma_k^{(i)}\right);(\xi_k^{(i)})}\left(z_1,...,z_r;u,a\right)$ is defined in (5.2.28).

Now if we apply (5.9.2), then the Theorem-1, 2 and 3 reduce to the following Corollaries- 4, 5 and 6 respectively related to the Liouville-Caputo fractional derivative.

**Corollary-4**

Let each of the parametric constraints $c,\gamma,v,\rho\in R^+$, $0<\sigma<1$ hold true. Suppose also that the general function $\in^{(\mu_r);(\delta_r)}_{(\alpha_r),\beta;(p_r)}\left(z_1,...,z_r;k,a\right)$ defined in (5.2.9) exists. If we set

$$\psi_0=N(0+)=N(t)\big|_{t=0} \qquad \text{...(5.9.10)}$$

then the solution of the following general hybrid type family of the fractional order kinetic equations

$$N(t)-N_0t^{\gamma-1}\in^{(\mu_r);(\delta_r)}_{(\alpha_r),\beta;(p_r)}\left(z_1t^{v_1},...,z_rt^{v_r};k,a\right)=-c^{\rho}\left({}^{LC}D^{\sigma}_{0+}N\right)(t) \qquad \text{...(5.9.11)}$$

is given by

$$N(t)=N_0t^{\gamma-1}\sum_{q=0}^{\infty}(-1)^q\left(\frac{t^{\sigma}}{c^{\rho}}\right)^{q+1}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)\left(a+\sum_{i=1}^{r}p_in_i\right)^k}$$

$$\frac{1}{\Gamma\left(\sum_{i=1}^{r}v_in_i+\sigma(q+1)+\gamma\right)}\prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_in_i}\left(z_it^{v_i}\right)^{n_i}}{n_i!}\right]+\psi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{\Gamma[\sigma q+1]}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^q$$

$$(t>0) \qquad \ldots(5.9.12)$$

provided that the right hand side of the solution (5.9.12) exists.

**Corollary-5**

Let each of the parametric constraints $c,\gamma,v,\rho\in R^{+}$, $0<\sigma<1$ hold true. Suppose also that the Multivariable Mittag-Leffler function $E_{(\alpha_r),\beta}^{(\mu_r);(\delta_r)}(z_1,\ldots,z_r)$ defined by the equation (5.2.24) exists. If $\psi_0$ is defined by the equation (5.9.10) then the solution of the following general hybrid type family of the fractional order kinetic equations

$$N(t)-N_0t^{\gamma-1}E_{(\alpha_r),\beta}^{(\mu_r),(\delta_r)}\left(z_1t^{v_1},\ldots,z_rt^{v_r}\right)=-c^{\rho}\left({}^{LC}D_{0+}^{\sigma}N\right)(t) \qquad \ldots(5.9.13)$$

is given by

$$N(t)=N_0t^{\gamma-1}\sum_{q=0}^{\infty}(-1)^q\left(\frac{t^{\sigma}}{C^{\rho}}\right)^{q+1}\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma\right)}{\Gamma\left(\beta+\sum_{i=1}^{r}\alpha_in_i\right)\Gamma\left(\sum_{i=1}^{r}v_in_i+\gamma+\sigma(q+1)\right)}$$

$$\times\prod_{i=1}^{r}\left[\frac{(\mu_i)_{\delta_in_i}\left(z_it^{v_i}\right)^{n_i}}{n_i!}\right]+\psi_0\sum_{q=0}^{\infty}\frac{(-1)^q}{\Gamma[\sigma q+1]}\left(\frac{t^{\sigma}}{c^{\rho}}\right)^q \qquad (t>0) \qquad \ldots(5.9.14)$$

provided that the right hand side of the solution (5.9.14) exists.

**Corollary-6**

Under the parametric constraints $c,\gamma,v_i,\rho\in R^{+}$ and $0<t<1$. Suppose that the extended multivariable Hurwitz-Lerch zeta function defined in (5.2.5) exists. If $\psi_0$ is defined by the equation (5.9.10), then the solution of the following generalized hybrid type fractional order kinetic equation

$$N(t)-N_0 t^{\gamma-1}\phi_{\theta_i;\left(\sigma_k^{(i)}\right),\left(\xi_k^{(i)}\right)}^{\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1 t^{v_1},\ldots,z_r t^{v_r};u,a\right)=-c^{\rho}\left({}^{LC}D_{0+}^{l}N\right)(t) \qquad \ldots(5.9.15)$$

is given by

$$N(t)=N_0 t^{\gamma-1}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t^l}{c^{\rho}}\right)^{m+1}\frac{\Gamma\gamma}{\Gamma\left(l(m+1)+\gamma\right)}\Phi_{\theta_i;(l(m+1)+\gamma);v_i,\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}^{(\gamma);v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1 t^{v_1},\ldots,z_r t^{v_r};u,a\right)$$

$$+\psi_0\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma[lm+1]}\left(\frac{t^l}{c^{\rho}}\right)^m \qquad (t>0) \qquad \ldots(5.9.16)$$

provided that the right hand side of the solution of the equation (5.9.16) exists and the function $\Phi_{\theta_i;\left(\sigma_k^{(i)}\right);\left(\xi_k^{(i)}\right)}^{\gamma;v_i,\left(\mu_j^{(i)}\right);\left(\delta_j^{(i)}\right)}\left(z_1,\ldots,z_r;u,a\right)$ is defined in (5.2.28).

We also mention the following known results as given below.

(i) At $r=1$ and $p_1=1$, Theorem-1 reduce to the known result due to Srivastava [94, p. 2658, Theorem 3.1 at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] for the function $\in_{\alpha,\beta}(\phi;z,s,a)$ considered therein [94].

(ii) At $r=1$, Theorem-2 reduce to the known result due to Srivastava [94, p. 2660, Theorem 3.3 at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] for Mittag-Leffler function.

(iii) At $r=1$ and for $\theta_1=1, p_1=p$ and $q_1=q$, Theorem-3 reduce to the known result due to Srivastava [94, p. 2661, Theorem 3.4].

(iv) At $r=1$ and $p_1=1$, Corrolary-1 and Corrolary-4 reduce to the known result due to Srivastava [94, p.2661, Corollary 4.2; p.2663, Corollary 4.6 at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] respectively for the function $\in_{\alpha,\beta}(\phi;z,s,a)$ considered therein [94].

(v) At $r=1$, Corollary-2 and Corollary-5 reduce to the known result due to Srivastava [94, p.2662, Corollary 4.2; p.2663, Corollary 4.7 at $\phi(n)=\frac{(\mu)_{\delta n}}{n!}$] for Mittag-Leffler function.

(vi) At $r=1$ and for $\theta_1=1, p_1=p$ and $q_1=q$, Corollary-3 and Corollary-6 reduce to the known result due to Srivastava [94, p.2662, Corollary 4.2; p.2664, Corollary 4.8].

## 5.10 SPECIAL REMARK

In present chapter, we find out the solution of general hybrid-type fractional order kinetic equation for the Liouville-Caputo fractional derivative operator $\left({}^{LC}D_{0+}^{u} f\right)(x)$ in section-A and for the Hilfer type fractional derivative operator $\left({}^{H}D_{0+}^{\sigma,\omega} f\right)(x)$ in section-B, involving the multivariable general function associated to multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta functions.

The main theorems of this chapter are very general and capable to be yielding solutions of a significantly large number of known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving Hurwitz-Lerch Zeta function of one and several variables.

Each of corollaries itself is significantly general in nature and can provide the solution of many relatively simple fractional order kinetic equations.

The new direction of research lies in the fact that the function $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}\left(z_1,...,\text{z}_r;s,a\right)$ can be further generalized for a sequence of parameter $\sigma\left(n_1,...,n_r\right)$ as follows:

$$\in_{(\alpha_r);\beta;(p_r)}^{\sigma}\left(z_1,...,z_r;s,a\right) = \sum_{n_1,...,n_r=0}^{\infty} \frac{\sigma\left(n_1,...,n_r\right) z_1^{n_1} ... \; z_r^{n_r}}{\left(a+\sum_{i=1}^{r} p_i n_i\right)^{s} \Gamma\left(\beta+\sum_{i=1}^{r} \alpha_i n_i\right)}$$

# CONCLUSION

# CONCLUSION

Here we are presenting the conclusion of the work which we have done throughout our study in a nice manner with the following points-

- I am submitting the thesis entitled- **“Investigations in certain-generalized special functions and fractional calculus operators with applications"** I have explained my work in five chapters. First chapter presents an introduction and historical overview of advancements in special functions and fractional calculus. This provides the definitions, symbols and notations which we have used in establishing new interesting results to develop the field of special function and fractional calculus.

- **Chapter-2** is organized into Section-A and B. A general function $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)$ and its mild generalization introduces in section-A and in Section-B, we extend this general function to a multivariable analogue for enhancing its applicability and scope. This general function unifies all the function related to Hurwitz-Lerch Zeta function, its extension as well as the Mittag-Leffler function and their extension scattered in the literature. Therefore, many known and new results are shown to be obtained for the Hurwitz-Lerch Zeta function as application of our main results.The results obtained in this chapter are related with the Hurwitz Lerch Zeta functions of single and multivariable that will be useful to solve a variety of problems of fractional kinetic equations, integral transforms, integral representations and generating functions.

- **Chapter-3** extends the analysis of general function. This chapter also structured into two section A and B. Some integral representation and fractional derivatives for general function $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)$ and further for its mild extension have established in section-A while in section-B we have obtained certain integral representation and expressions for multivariable general function $\in_{(\alpha_r),\beta;(p_r)}^{(\mu_r);(\delta_r)}(z_1,...,\mathrm{z}_r;s,a)$ and similar results are further established for its mild extension.

We have evaluated certain type of integrals involving the Mittag-Leffler function which gives the general function as resultant for single and multivariable. General function studied in this chapter unifies all the function related to Hurwitz-Lerch Zeta function, its extension as well as the Mittag-Leffler function and their extension. Therefore, certain integral expressions and fractional derivative related to Hurwitz-Lerch Zeta function and the simpler functions are shown to be derived.

Due to the general nature of the general functions, the results obtained in this chapter will be very helpful and provide new direction and further scope of research for the young researchers who are actively engaged in the field.

- **Chapter-4** is presented into the section-A and B. In section-A we have established six fractional integral and derivative formulae for Marichev-Saigo-Maeda fractional integral and derivative operators, and Caputo type Marichev-Saigo-Maeda fractional derivative operators involving extended k-generalized Mittag-Leffler function $E_{k,l,m}^{\rho,\sigma;c}(x;p)$. In section-B, we have considered extended k-generalized Mittag-Leffler type function $E_{\alpha,\beta}^{\rho,c;j,\gamma}(z;p)$ and six fractional integral and derivative formulae for Marichev-Saigo-Maeda fractional integral and derivative operators and Caputo type Marichev-Saigo-Maeda fractional derivative operators involving this function are obtained.

The fractional order differentiation and integration of special functions are very important as these results are being used to solve the fractional order differential and integral equations. Due to the applications of fractional calculus, the present study is very much significant for the field of science and engineering.

In this chapter the generalized fractional calculus operators are used to evaluate the fractional differential and integrals formulae. The importance of the results established in this chapter lies in the fact that the extended k-generalized Mittag-Leffler function and extended k-generalized Mittag-Leffler type function are very general in the nature therefore many known and new results can be obtained for three and four parameter Mittag-Leffler type functions. Main results of this chapter are capable to provide the fractional integral and differential formulae related to Riemann-Liouville and Saigo operators. These results are recorded in the corresponding corollaries.

- In **chapter-5** we have obtained the solution of general hybrid type fractional order kinetic equations with fractional derivative operators using Laplace transform method. This chapter is also divided into two sections A and B.

  We have derived solutions of general hybrid-type fractional-order kinetic equation for Liouville-Caputo fractional derivative operator in Section A and for Hilfer-type fractional derivative operator in Section B, associated to multivariable general function associated to multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta functions.

  The main theorems of this chapter are very general and capable to be yielding solutions of a significantly large number of known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving Hurwitz-Lerch Zeta function of one and several variables. New results for the fractional-order derivative operator are presented as corollaries. Each of corollaries itself is significantly general in nature and can provide the solution of many relatively simple fractional order kinetic equations.

  The work we have done in these Chapter-2,3,4 and 5 achieved our objectives. Hence all Theorems and Corollaries established in this thesis are new and significant. Therefore, this study will stimulate further progress in the fields of special functions and fractional calculus.

**Further scope of research**: The methodology and approaches that we have used in the thesis to develop the results can be further extended for more general new special functions and fractional operators. The young researchers will utilize our finding to develop new study involving the Marichev-Saigo-Maeda fractional integral and derivative, Caputo-Marichev-Saigo-Maeda fractional derivative, Caputo derivative, Hilfer derivative, general hybrid type fractional order kinetic equation and for more general multivariable multi-index general function, Hurwitz -Lerch zeta function and k-Mittag- Leffler function, to solve the complicated problems of applied mathematics, science and engineering.

# SUMMARY

# SUMMARY

The thesis entitled **"Investigations in Certain-Generalized Special Functions and Fractional Calculus Operators with Applications"** is divided into five chapters. First chapter presents an introduction and includes the definitions, symbols and notations which we have used in establishing new interesting results to develop the field of special function and fractional calculus.

The remaining four chapters-2,3,4,5 exhibit our main research work. There after we have written the conclusion of our research work done in these chapters. Complete bibliography of the present study for reference is displayed at the end of the thesis. Two research papers on these four chapters are published while six are communicated for publication. Here in this section now we provide the summary of thesis of the research work done in all these five chapters.

**Chapter 1: Introduction and preliminaries to the field of study**

We initiated this thesis with Chapter first, it is preliminary and presents the brief historical advancements of the related field of special functions and fractional calculus. The chapter has eight sections- 1.1 to 1.8. After introducing the field of research in the section-1.1, we have given the definitions of hypergeometric function of single and multi-variables like Gauss hypergeometric function, confluent hypergeometric function, generalized hypergeometric function, Wright's generalized hypergeometric function, Kampe de Ferriet function, Hurwitz-Lerch Zeta functions of single and multivariable and its generalization, Mittag-Leffler function of single and multivariables, definition of elementary Gamma and Beta functions and their k-generalization, k-Mittag-Leffler type function of arbitrary order and Barnes class of functions in section-1.2. Linear generating functions, multivariable and multilinear generating functions are also defined in section 1.3.

Fractional Calculus or the Calculus of arbitrary order is also an important field of research. We have done some work on the application of fractional calculus in this thesis. In section-1.4, the fractional Calculus, the Riemann-Liouville fractional integral operators, differential operators are defined further the Saigo operator, Erdelyi-Kober operators, Marichev-Saigo-Maeda operators and their Caputo-type generalization and Hilfer derivative operators are also defined. We have defined the Laplace transform, Inverse Laplace transform and its convolution property and Laplace transform of derivative are given in section - 1.5. The generalized fractional kinetic equations are studied in this thesis therefore we have presented the elementary knowledge on the kinetic equations in section-1.6. The glossary of symbols used in the thesis is given in section-1.7 and the abstract of remaining four chapters- 2, 3, 4, 5 is given in brief in section-1.8.

**Chapter 2: Generating functions involving general special function related to Hurwitz-Lerch Zeta function and its multivariable generalization**

This chapter is divided into two sections A and B and fourteen subsections 2.1-2.14. In subsection-2.1 is chapter introduction is given while the in subsection-2.2 we have given the definitions of Hurwitz-Lerch Zeta function and Mittag-Leffler function with their extensions and related functions. A single variable general function which unifies the Hurwitz-Lerch Zeta function and Mittag-Leffler function is also defined here.

Section-A deals with a general function of single variable. It included five subsection 2.3 to 2.7. An introduction of section-A has given in 2.3. We have given the integral representation for general function with its proof in subsection-2.4. The four theorems on generating functions have given with their outline of proofs in subsection-2.5. A mild extension of this general function has also considered and the integral representation and four theorems on generating function for this mild extension are also given with their proofs in subsection-2.6. As applications of our main theorems in subsection-2.7, we have obtained six known results associated to Hurwitz-Lerch Zeta function deduced from these main results of section-A.

In Section-B we have studied a multivariable analogue of general function defined in Section-A. This section further divided into subsections from 2.8 to 2.14. We have considered multivariable general function in subsections 2.8 and in 2.9 we have given the definitions related to multivariable Hurwitz-Lerch Zeta function and Mittag-Leffler function. The integral representation for multivariable general function with its proofs is given in subsection-2.10. The four theorems on generating functions for this function have given with their proofs in subsection-2.11. One theorem on the integral representation and four theorems on the generating function for its mild extension are also given with their proofs in subsection-2.12. As applications of our main theorems, we have given known and new results associated to Hurwitz-Lerch Zeta function, the new results of main theorems are given as corollaries in subsection-2.13. In subsection-2.14, the special remark is given on the recent work done.

**Chapter-3: Integral representations and integral expressions associated to the general functions of one and several variables in terms of Mittag-Leffler functions**

The motive of this chapter is to evaluate certain integral representations, fractional derivatives and integral expressions for single and multivariable general function in terms of Mittag-Leffler function of single and multivariable. This chapter is divided into section-A and B with fourteen subsections 3.1 to 3.14. In first two subsections-3.1 and 3.2 of this chapter we have given introduction to the chapter and the definitions of fractional derivative, Hurwitz-Lerch Zeta function, Mittag-Leffler function and their extensions and integral representation of general function, some elementary formulae are also given their.

In section-A, we have included five subsections-3.3 to 3.7. In subsections-3.3 and 3.4, we have given three main theorems on integral representation and one theorem on fractional derivative for general function of single variable with their proofs. In subsection-3.5 we have considered a mild extension of general function in view of parameters and given the integral representation this function. Three theorems on integral representations and one theorem on fractional derivative for this mild extension with their proofs have given in subsection-3.6. In subsection-3.7 we have given some known special cases related to Hurwitz-Lerch Zeta function as the applications of our main results.

Section-B is further extended to subsection-3.8 to 3.14. In first subsections-3.8, the multivariable analogue of general function studied in section-A is defined in subsection-3.9, we have given the definitions of multivariable Hurwitz-Lerch Zeta function, multivariable Mittag-Leffler function and one theorem on integral representation for this multivariable general function with the proof also given here. Three theorems on the integral expressions are given in subsection-3.10 in terms of Mittag-Leffler function for multivariable general function have given with their proofs. In subsections-3.11 and 3.12 we have considered a mild extension for this multivariable general function and also a new extended Mittag-Leffler multivariable function therein. The four related theorems on integral expressions for this extended general function and their proofs have given in subsection-3.13. In the subsection 3.14, we have given the new and known results deduced from main results and some new results have recorded here as corollaries.

**Chapter-4: Generalized fractional integral and differential operators and certain extended k- generalized Mittag-Leffler type functions**

This chapter is devoted to the new study of generalized fractional calculus for extended k-generalized Mittag-Leffler function and its extended k-generalized Mittag-Leffler type function via Marichev-Saigo-Maeda integral and differential operators. The Caputo-type Marichev-Saigo-Maeda fractional derivatives are also established for extended k-generalized Mittag-Leffler function as well as for extended k-generalized Mittag-Leffler type function. The chapter have two parts: Section-A and B with ten subsections-4.1 to 4.10.

The first two subsections-4.1 and 4.2 of this chapter have given definitions of Riemann-Liouville, Marichev-Saigo-Maeda fractional integrals and derivative operators Erdelyi-Kober operators, Caputo-type Marichev-Saigo-Maeda fractional derivative operators, the Mittag-Leffler function and its k-generalization, the k-Beta function, k-Gamma function and related function are also given as preliminaries.

Section-A, presents six fractional integral and differentiation formulae for extended k-generalized Mittag-Leffler function using the generalized Marichev-Saigo-Maeda fractional integral and derivative operators. It includes three subsections-4.3 to 4.5. In subsections-4.3 and 4.4, we have established six theorems using Marichev-Saigo-Maeda operators, two of them deal with the Marichev-Saigo-Maeda fractional integration. While next two theorems deal with the Marichev-Saigo-Maeda fractional differentiation and the remaining two theorems are associated with the Caputo-type Marichev-Saigo-Maeda fractional derivatives for the extended k-generalized Mittag-Leffler function with their proofs. In subsection 4.5, we have given the Riemann-Liouville integral and fractional derivative, Erdelyi-Kober fractional integral and derivative, Saigo and Caputo fractional integral and fractional derivative of extended k-generalized Mittag-Leffler function, as the applications of the main theorems the twelve corollaries are given. Certain known results for simpler functions are also shown to be obtained here.

Section-B have extended to subsections-4.6 to 4.9. In the first two subsection-4.6 and 4.7, we have studied a new k-extended Mittag-Leffler type function of arbitrary order. The definitions of related functions of arbitrary order are also given here. In next subsection-4.8, we have given six theorems on generalized fractional operators involving the extended k-generalized Mittag-Leffler type function with their proofs. In subsection-4.9, as applications of the main theorems, the Riemann-Liouville integral and fractional derivative, Erdelyi-Kober fractional integral and derivative, Saigo and Caputo fractional integral and fractional derivative involving this extended k-generalized Mittag-Leffler type function have shown to be obtained and given in twelve corollaries. Certain known results for simpler form of Mittag-Leffler functions are also obtained here in this subsection. In subsection-4.10, we have mentioned some known special cases for the functions studied by some earlier workers.

**Chapter-5: Fractional calculus operators and certain fractional order kinetic equations and differential equations with their solutions**

In this chapter, we have obtained the solution of a general family of hybrid-type fractional order kinetic equation for fractional derivative operators involving multivariable general function associated to the multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta functions using the Laplace transform method. This chapters divided into two sections A and B. This chapter includes ten subsections. In first two subsectios-5.1 and 5.2, we have given the definitions of Hurwitz-Lerch Zeta function, Mittag-Leffler function and their extensions. The multivariable general function is also considered here. The operators Riemann-Liouville, Liouville-Caputo and the kinetic equations and Laplace transform with related functions are also defined. Certain Laplace transform formulae related to these multivariable functions are also given for the necessity to establish the main theorems.

Section-A have included 3 subsections-5.3 to 5.5. In subsections 5.3 and 5.4, we have given three theorems of general hybrid-type fractional order kinetic equations involving multivariable general function associated to the multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta functions for Liouville-Caputo fractional derivative operator with their proofs. Our main results are very general and capable to be yielding solutions of many known fractional order kinetic equations involving the Mittag-Leffler function as well as involving Hurwitz-Lerch Zeta function of one and several variables, certain special cases have given in subsection-5.5.

Section-B divided into five subsections-5.5 to 5.10. In first two subsections 5.6 and 5.7, we have given the definitions of Hilfer fractional derivative and Laplace transform of Hilfer derivative. In subsection 5.8, we have given three theorems on general hybrid-type fractional order kinetic equations involving multivariable general function associated to the multivariable Mittag-Leffler function and multivariable Hurwitz-Lerch Zeta functions for Hilfer fractional derivative operator with their proofs. In subsection-5.9, new results for the Riemann-Liouville and Liouville-Caputo fractional-order derivative operator have obtained in six corollaries and certain known results also shown to be derived here.

In subsection-5.10 we have given a special remark as the main results studied here are very general and capable to be yielding solutions of many known and new fractional order kinetic equations involving the Mittag-Leffler function as well as involving Hurwitz-Lerch Zeta function of one and several variables. The new direction of research about the unified definition of the general function studied in this chapter is given in view of general sequence of parameters.

For the references used in all these five chapters, a complete bibliography of 116 references is given at the end of this thesis.

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# ANNEXURE-1

## RESEARCH PAPER CONTRIBUTED BY THE AUTHOR

The main results of all papers of the chapters of the present Thesis "**Investigations in Certain-Generalized Special Functions and Fractional Calculus Operators with Applications**" have been published/communicated for publication. The following is the chapter wise list of research papers contributed by the author having a bearing on the subject matter of the present thesis.

**Chapter-2**

1. Generating Function Involving General Function Related to Hurwitz-Lerch Zeta function. By B.B.Jaimini and M.K.Tatwal; published in Communications on Applied Nonlinear Analysis Vol. 32, No. 8s (2025), 951-962, ISSN No.1074-133X.

2. Multivariable extension of general class of functions associated with the Hurwitz-Lerch Zeta function and certain generating relations.
(Communicated for publication)

**Chapter-3**

1. Certain integral representation and fractional derivatives associated to a general function related to Hurwitz-Lerch Zeta function. By B.B.Jaimini and M.K.Tatwal; published in Panamerican Mathematical Journal Vol. 35, No. 4s (2025), 1240-1250, ISSN No.1064-9735.

2. Multivariable analogue of a function associated with the Hurwitz-Lerch Zeta function and its integral expressions in term of Mittag- Leffler functions of several variables.(Communicated for publication)

**Chapter-4**

1. The generalized fractional calculus of the extended k-generalized Mittag-Leffler function. (Communicated for publication)

2. On extended k-generalized Mittag-Leffler type function and its generalized fractional integration and differentiation.(Communicated for publication)

**Chapter-5**

1. The general hybrid-type fractional order kinetic equation involving general multivariable function for Riemann-Caputo fractional derivative operator. (Communicated for publication)

2. Solution of general hybrid-type fractional order kinetic equations associated with the multivariable general function for Hilfer type fractional derivative operator. (Communicated for publication)

# ANNEXURE-2

## RESEARCH PAPER PUBLISHED/PRESENTED

The present Thesis "**Investigations in Certain-Generalized Special Functions and Fractional Calculus Operators with Applications**" consist of the following published/presented research papers.

**(i) List of Publications.** (Reprints Attached)

1. Generating Function Involving General Function Related to Hurwitz-Lerch Zeta function. By B.B.Jaimini and M.K.Tatwal; published in Communications on Applied Nonlinear Analysis Vol. 32, No. 8s (2025), 951-962, ISSN No.1074-133X.
2. Certain integral representation and fractional derivatives associated to a general function related to Hurwitz-Lerch Zeta function. By B.B.Jaimini and M.K.Tatwal; published in Panamerican Mathematical Journal Vol. 35, No. 4s (2025), 1240-1250, ISSN No.1064-9735.

**(ii) List of Papers presented in Conference/Seminar.** (Certificate Attached)

1. "Fractional Calculus Integral Operator Associated with Generalized Special Functions" Presented in International Conference on Multidisciplinary Approach to Navigating Emerging Trends & Challenges in Global Commerce, Humanities and Science for Sustainable Development held on 13-14 January, 2025 organized by Seth Nand Kishor Patwari Government College Neem Ka Thana (Rajasthan), India.
2. "Certain Integral Representation and Fractional Derivatives Associated to Generalized Special Function Related to Hurwitz-Lerch Zeta Function" Presented in International Multidisciplinary Conference on Recent Advancements in Education, Artificial Intelligence Humanities, Social Sciences & Sciences for Achieving Sustainable Development Goals held on 29-30 August, 2025 organized by Vivek P.G. Mahavidhyalaya, Kalwar, Jaipur and Inspira Research Association-IRA, Jaipur (Rajasthan), India.
3. "Certain results involving Multivariable class of functions associated with extended Hurwitz Lerch Zeta Function" Presented in 3rd International Conference

on Computational Applied Sciences & It's Application (ICCASA-2025) held on 20-21 September, 2025 organized by University of Engineering & Management, Jaipur (Rajasthan), India.

# PUBLISHED RESEARCH PAPERS IN JOURNALS

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 8s (2025)

# Generating Function Involving General Function Related to Hurwitz-Lerch Zeta Function

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***Article History:***
***Received:*** *01-01-2025*
***Revised:*** *22-02-2025*
***Accepted:*** *27-02-2025*

**Abstract:**
In this paper, we have studied a general function which unifies the Hurwitz-Lerch Zeta function and Mittage-Leffler function. The integral representation of the function and certain generating functions involving this general function are established. A mild extension of this general function is also presented in this paper. The generating functions for this extended function are also studied in this paper. Certain known results involving Hurwitz-Lerch Zeta function for several parameters are shown to be obtained here.

**Keywords**: Hurwitz-Lerch Zeta function; Mittag-Leffler function; Generating functions; Generalized Hypergeometric function; Integral representation

**MSC 2020**: Primary 11M35; 33E20 Secondary: 33E12; 33C20

## 1. Introduction and Preliminaries

A general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in the literature in the following manner [1]:

$$\phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \qquad \ldots(1.1)$$

$$(a\in C/Z_0^-, s\in C \text{ when } |z|<1 \text{ and } \mathrm{Re}(s)>1 \text{ when } |z|=1)$$

Where $C,R,R^+,Z$ represents the Set of complex numbers, Set of real numbers, Set of positive real numbers and Set of integer number respectively and $Z_0^-=Z^-\cup\{0\},\ Z^-=\{-1,-2,-3...\}$

The Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in (1.1) contains, its special cases, as the Riemann Zeta function $\zeta(s)$ and Hurwitz-Lerch Zeta function $\zeta(s,a)$ defined by (see, for details,[1, chapter I])

$$\zeta(s)=\sum_{n=0}^{\infty}\frac{1}{(n+1)^s}=\phi(1,s,1)\ \left(\mathrm{Re}(s)>1\right) \qquad \ldots(1.2)$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^{s}}=\phi(1,s,a)\qquad\left(\operatorname{Re}(s)>1;a\in C/Z_0^-\right)\qquad\ldots(1.3)$$

A generalization of the Hurwitz-Lerch Zeta function $\phi(z,s,a)$ was studied by Lin and Srivastava [2, p.727, eq.(8)]

$$\phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^{n}}{(n+a)^{s}}\qquad\ldots(1.4)$$

($\mu\in C;a,\nu\in C/Z_0^-,\rho,\sigma\in R^{+};\rho<\sigma$ when $s,z\in C;\rho=\sigma$ and $s\in C$ when $|z|<\delta=\rho^{-\rho}\sigma^{\sigma};\rho=\sigma$ and $\operatorname{Re}(s-\mu-\nu)>1$ when $|z|=\delta$)

Where $(\lambda)_n$ denotes the general pochhammer symbol which is defined by in terms of the gamma function by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma\lambda}=\begin{cases}1 & \left(n=0;\lambda\in C/\{0\}\right)\\ \lambda(\lambda+1)\ldots\lambda(\lambda+n-1) & (n=N;\lambda\in C)\end{cases}\qquad\ldots(1.5)$$

The function at $\rho=\sigma=1$ and $\nu=1$ yields the Hurwitz-Lerch Zeta function defined and studied by Goyal and Loddha [3, p.100, eq.(1.5)]

$$\phi_{\mu}^{*}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^{n}}{(n+a)^{s}}\qquad\ldots(1.6)$$

($\mu\in C;a\in C/Z_0^-,s\in C$ when $|z|<1$ and $\operatorname{Re}(s-\mu)>1$ when $|z|=1$)

Further generalization of the functions $\phi(z,s,a)$ and $\phi_{\mu}^{*}(z,s,a)$ was considered by Garg et.al [4] in the following manner [4,p.313, eq.(1.7)]:

$$\phi_{\lambda,\mu,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n\, n!}\frac{z^{n}}{(a+n)^{s}}\qquad\ldots(1.7)$$

($\lambda,\mu\in C;\nu,a\in C/Z_0^-;s\in C$ when $|z|<1$ and $\operatorname{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=1$)

A further generalization of a family of Hurwitz-Lerch Zeta function defined and studied by Srivastava et al [5, p.491, eq.(1.20)] in the following form:

$$\phi_{\lambda,\mu;\nu}^{(\rho,\sigma,k)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}\,n!}\frac{z^n}{(n+a)^s} \qquad \ldots(1.8)$$

$(\lambda,\mu\in C;\nu,a\in C/Z_0^-;\rho,\sigma,\mathrm{k}\in\mathrm{R}^+;k-\rho-\sigma>-1$ when $s,z\in C;k-\rho-\sigma=-1$ and $s\in C$ when $|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}\mathrm{k}^k;k-\rho-\sigma=-1$ and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=\delta^*)$

The well-known Mittage-Leffler function $E_\alpha(z)$ in defined by following manner [13](See also [14,15])

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ (z\in C,\mathrm{Re}(\alpha)>0) \qquad \ldots(1.9)$$

This function further generalized by Prabhakar[16] in the following form [See also [17]]:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!} \qquad \ldots(1.10)$$

$$(\alpha,\beta,\gamma\in C;\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0)$$

In view of extension of parameters, Mittage-Leffler function was also studied by Khan et al [6, p.2, eq.(1.9)]

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\eta,\mathrm{q}}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}(\delta)_{pn}}\frac{z^n}{\Gamma(\alpha n+\beta)} \qquad \ldots(1.11)$$

Where $\alpha,\beta,\eta,\delta,\mu,\nu,\rho,\sigma\in C;p,q>0$ and $\mathrm{q}\le\mathrm{Re}(\alpha)+p$ and,
$\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\delta),\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$ if $\delta=p=1$, it takes the following form

$$E_{\alpha,\beta,\nu,\sigma}^{\mu,\rho,\eta,\mathrm{q}}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(1.12)$$

Where $\alpha,\beta,\eta,\mu,\nu,\rho,\sigma\in C;p,q>0$ and $\mathrm{q}\le\mathrm{Re}(\alpha)$ and,
$\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$

Wright Introduced and studied the Taylor-Maclaurin Series [7, Pg.:424]:

$$\in_{\alpha,\beta}(\phi;z)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{\Gamma(\alpha n+\beta)} \quad (\alpha,\beta\in C;\mathrm{Re}(\alpha)>0) \qquad \ldots(1.13)$$

Where $\phi(n)$ is a function satisfying suitable conditions Motivated by the work of Wright[7] E.W. Barnes [23] considered the asymptotic expansion of functions in the class which is defined as follows:

$$E_{\alpha,\beta}^{(k)}(s;z)=\sum_{n=0}^{\infty}\frac{z^n}{(n+k)^s\Gamma(\alpha n+\beta)}\quad(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0)\qquad\ldots(1.14)$$

For suitable restricted parameters $k$ and $s$

Recently for suitably restricted sequence $\{\phi(n)\}_{n=0}^{\infty}$, Srivastava [8] introduced and studied a class of function [See also [9][10]]:

$$\in_{\alpha,\beta}(\phi;z,s,a)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{(a+n)^s\Gamma(\alpha n+\beta)}\quad\text{where}(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0)\qquad\ldots(1.15)$$

For $\phi(n)=\frac{(\mu)_n}{n!}$ the following function recently studied by Virendra [11, p.13, Eq.(1.9)]:

$$\in_{\alpha,\beta,\delta}^{\mu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(\delta n+a)^s\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\qquad\ldots(1.16)$$

Where $\mathrm{Re}(\mu)>0;\mathrm{Re}(\delta)>0;\mathrm{Re}(a)>0;\mathrm{Re}(s)\geq 0;\mathrm{Re}(\alpha)>0;\mathrm{Re}(\beta)>0$ and $|z|\leq 1$

To give more extension in view of the parameters, we consider here the following more generalized function defined and represented as below.

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}z^n}{(\sigma)_{\xi n}(a+n)^s\Gamma(\alpha n+\beta)n!}\qquad\ldots(1.17)$$

Where $(\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_0^-;\delta,\nu,\xi\in R^+;\xi-\delta-\nu>-1$ when $s,z\in C;\xi-\delta-\nu=-1$ and $s\in C$ when $|z|<\rho^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi};\xi-\delta-\nu=-1$ and $\mathrm{Re}(s+\sigma-\mu-\lambda)>1$ when $|z|=\rho^*)$

The integral representation of function defined in (1.17) is also obtained here and given as the following theorem.

**Theorem-1.1**

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_0^-;\delta,\nu,\xi\in R^+;\xi-\delta-\nu\geq-1$ with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$; integral representation of the function defined in (1.17) $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)$ holds true

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}e^{-at}t^{s-1}\left\{E_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(ze^{-t})\right\}dt\qquad\ldots(1.18)$$

Where $E_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittage-Leffler function defined in (1.12)

Proof: The function defined in (1.17) can be written straight in the following form

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\xi n}n!\Gamma(\alpha n+\beta)}\left\{\frac{\Gamma s}{(a+n)^{s}}\right\}$$

Now using the following elementary Gamma integral

$$\frac{\Gamma s}{(a+n)^{s}}=\int_{0}^{\infty}t^{s-1}e^{-(a+n)t}dt \qquad \left(\min\{\operatorname{Re}(s),\operatorname{Re}(a)\}>0; n\in N_{0}\right) \qquad \ldots(1.19)$$

and then changing the order of integration and summation, we have

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\int_{0}^{\infty}e^{-at}t^{s-1}\left\{\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\xi n}\Gamma(\alpha n+\beta)}\frac{(ze^{-t})^{n}}{n!}\right\}dt$$

On interpreting the inner series in view of (1.10) we at once arrive at the desired result it completes the proof of theorem (1.1).

Throughout the paper $\left\{\begin{matrix} m+l-1 \\ m \end{matrix}\right\}=\frac{(m)_{l}}{l!}$ where $(m)_{l}$ is the pochhammer's notation

## 2. Generating Functions

The four generating functions involving the general function define in (1.17) are establish here.

**Theorem-2.1**

For $\min\{\operatorname{Re}(a),\operatorname{Re}(s)\}>0; \alpha,\beta,\mu,\lambda\in C; a,\sigma\in C/Z_{0}^{-}; \delta,\nu,\xi\in R^{+}; \xi-\delta-\nu\geq-1$ with $|z|<\delta^{*}=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$ we have

$$\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+l,a)t^{l}=\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a-t) \qquad \ldots(2.1)$$

**Theorem-2.2**

For $\min\{\operatorname{Re}(a),\operatorname{Re}(s)\}>0; \alpha,\beta,\mu,\lambda\in C; a,\sigma\in C/Z_{0}^{-}; \delta,\nu,\xi\in R^{+}; \xi-\delta-\nu\geq-1$ with $|z|<\delta^{*}=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$ we have

$$\sum_{l=0}^{\infty}\binom{s+2l-1}{2l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+2l,a)t^{2l}=\frac{1}{2}\left[\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a+t)+\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a-t)\right] \qquad \ldots(2.2)$$

**Theorem-2.3**

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_0^-;\delta,\nu,\xi\in R^+;\xi-\delta-\nu\geq-1$ with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$ we have

$$\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+2l+1,a)t^{2l+1}=\frac{1}{2}\left[\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a-t)-\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a+t)\right] \quad \ldots(2.3)$$

**Theorem-2.4**

For $\mathrm{Re}\left(\sum_{i=1}^{p}\theta_i\right)>\mathrm{Re}\left(\sum_{j=1}^{q}\beta_j\right)>0;\ \left|\frac{t}{a}\right|<1$

$$\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j+l,a\right)\frac{t^l}{l!}$$

$$=\sum\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\xi n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\frac{1}{(a+n)^{\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j}}\,{}_pF_q\left[\begin{matrix}(\theta_p);\ \dfrac{t}{a+n}\\(\beta_q);\end{matrix}\right] \quad \ldots(2.4)$$

Where ${}_pF_q$ is a generalized hypergeometric function defined by

$${}_pF_q\left[\begin{matrix}(a_p); & \\ (b_q); & \end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!};\qquad (|z|<1) \quad \ldots(2.5)$$

**Outlines Of Proofs**

**Proof of Theorem 2.1:**

Let left hand side of (2.1) is denoted by $\Delta_1$ i.e.

$$\Delta_1=\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+l,a)t^l$$

On using the definition (1.17) and changing the order of summation we have

$$\Delta_1=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}z^n}{(\sigma)_{\xi n}(a+n)^s\Gamma(\alpha n+\beta)n!}\left\{\sum_{l=0}^{\infty}\frac{(s)_l}{l!}\left(\frac{t}{a+n}\right)^l\right\}$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
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Now on using the binomial expansion $(1-x)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n$, we have

$$\Delta_1=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}}{(\sigma)_{\xi n}\,n!\,\Gamma(\alpha n+\beta)}\frac{1}{(a-t+n)^s}$$

Now on interpreting the resulting series in view of (1.17) arrive at the desired result in (2.1)

**Proof of Theorem 2.2:**

Let left hand side of (2.2) is denoted by $\Delta_2$ i.e

$$\Delta_2=\sum_{l=0}^{\infty}\binom{s+2l-1}{2l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,v}(z,s+2l,a)t^{2l}$$

On using the definition (1.17) and changing the order of summation we have

$$\Delta_2=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}z^n}{(\sigma)_{\xi n}(a+n)^s\,\Gamma(\alpha+n\beta)n!}\left\{\sum_{l=0}^{\infty}\frac{(s)_{2l}}{(2l)!}\left(\frac{t}{a+n}\right)^{2l}\right\}$$

Now in view of the binomial expansion

$$\sum_{n=0}^{\infty}\frac{(a)_{2n}}{(2n)!}x^{2n}=\frac{1}{2}\left[(1-x)^{-a}+(1+x)^{-a}\right]$$

It takes the following form

$$\Delta_2=\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}z^n}{(\sigma)_{\xi n}\,n!\,\Gamma(\alpha n+\beta)(a-t+n)^s}+\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}z^n}{(\sigma)_{\xi n}\,n!\,\Gamma(\alpha n+\beta)(a+t+n)^s}\right]$$

On interpreting the resulting series in view of definition (1.17), we at once arrive at the desired result in (2.2)

**Proof of Theorem 2.3:**

Let left hand side of (2.3) is denoted by $\Delta_3$ i.e

$$\Delta_3=\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,v}(z,s+2l+1,a)t^{2l+1}$$

On using the definition (1.17) and changing the order of summation we have

$$\Delta_3=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}z^n}{(\sigma)_{\xi n}\,n!\,\Gamma(\alpha n+\beta)(a+n)^s}\left\{\sum_{l=0}^{\infty}\frac{(s)_{2l+1}}{(2l+1)!}\left(\frac{t}{a+n}\right)^{2l+1}\right\}$$

Now in view of the binomial expansion

$$\sum_{n=0}^{\infty}\frac{(a)_{2n+1}}{(2n+1)!}x^{2n+1}=\frac{1}{2}\left[(1-x)^{-a}-(1+x)^{-a}\right]$$

It takes the following form

$$\Delta_3=\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}z^n}{(\sigma)_{\zeta n}\Gamma(\alpha n+\beta)(a-t+n)^s n!}-\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}z^n}{(\sigma)_{\zeta n}\Gamma(\alpha n+\beta)(a+t+n)^s n!}\right]$$

On interpreting the inner series in view of definition (1.17), we at once arrive at the desired result in (2.3)

**Proof of Theorem 2.4:**

Let left hand side of (2.4) is denoted by $\Delta_4$ i.e,

$$\Delta_4=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\in_{\alpha,\beta,\sigma,\zeta}^{\mu,\delta,\lambda,v}\left(z,\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j+l,a\right)\frac{t^l}{l!}$$

On using the definition (1.17) and changing the order of summation we have

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}z^n}{(\sigma)_{\zeta n}n!\Gamma(\alpha n+\beta)(a+n)^{\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j}}\left\{\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\left(\frac{t}{a+n}\right)^l\frac{1}{l!}\right\}$$

Now interpreting the inner series into generalized hypergeometric function ${}_pF_q$ in view of (2.5) at once arrive at the desired result in (2.4)

## 3. Mild Extension And Associated Generating Functions

A mild extension is also considered here, it is defined and represented in the following manner:

$$\in_{\alpha,\beta,(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(a+n)^s\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\qquad\ldots(3.1)$$

Where $(p,q\in N_0;\ s,z,\mu_i,\alpha,\beta\in C\ (i=1,2,\ldots\ldots.\text{p});a,\sigma_j\in C/Z_0^-\ (j=1,2,\ldots\ldots.\text{q});\delta_i,\rho_j\in R^+$ $(i=1,2,\ldots\ldots.\text{p};j=1,2,\ldots\ldots.\text{q}))$

The integral representation for the above function defined in (3.1) is given in the following theorem.

**Theorem 3.1:**

For $p,q \in N_0; s,z,\mu_i,\alpha,\beta \in C\,(i=1,2,.......\text{p});\;\; a,\sigma_j \in C/Z_0^-\,(j=1,2,.......\text{q}); \delta_i,\rho_j \in R^+$
$(i=1,2,.......\text{p}; j=1,2,.......\text{q})$

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt \quad \text{...(3.2)}$$

Where $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is the Mittage-Leffler function defined as follows:

$$E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)=\sum_{n=0}^\infty \frac{\prod_{i=1}^p(\mu_i)_{\delta_i n}}{\prod_{j=1}^q(\sigma_j)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \quad \text{...(3.3)}$$

Proof: Theorem 3.1 can be easily proved following the similar lines as to prove Theorem 1.1

The four generating functions involving general function defined in (3.1) are establish here and given in the following theorem (3.2) to (3.5).

**Theorem 3.2:**

For $p,q \in N_0; s,z,\mu_i,\alpha,\beta \in C\,(i=1,2,.......\text{p}); a,\sigma_j \in C/Z_0^-\,(j=1,2,.......\text{q}); \delta_i,\rho_j \in R^+$
$(i=1,2,.......\text{p}; j=1,2,.......\text{q})$, we have

$$\sum_{l=0}^\infty \binom{s+l-1}{l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s}+1,a)t^l=\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s},a-\text{t}) \quad \text{...(3.4)}$$

**Theorem 3.3:**

For $p,q \in N_0; s,z,\mu_i,\alpha,\beta \in C\,(i=1,2,.......\text{p}); a,\sigma_j \in C/Z_0^-\,(j=1,2,.......\text{q}); \delta_i,\rho_j \in R^+$
$(i=1,2,.......\text{p}; j=1,2,.......\text{q})$, we have

$$\sum_{l=0}^\infty \binom{s+2l-1}{2l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s}+2\text{l},a)t^{2l}=\frac{1}{2}\left[\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s},a+\text{t})+\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s},a-\text{t})\right]\text{...(3.5)}$$

**Theorem 3.4:**

For $p,q\in N_0;s,z,\mu_i,\alpha,\beta\in C\left(i=1,2,.......\text{p}\right);a,\sigma_j\in C/Z_0^-\ \left(j=1,2,.......\text{q}\right);\delta_i,\rho_j\in R^+$
$\left(i=1,2,.......\text{p};j=1,2,.......\text{q}\right)$

$$\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\text{s}+2\text{l}+1,a\right)t^{2l+1}=\frac{1}{2}\left[\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\text{s},a-\text{t}\right)-\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\text{s},a+\text{t}\right)\right]\quad ...(3.6)$$

**Theorem 3.5:**

For $p,q\in N_0;s,z,\mu_i,\alpha,\beta\in C\left(i=1,2,.......\text{p}\right);a,\sigma_j\in C/Z_0^-\ \left(j=1,2,.......\text{q}\right);\delta_i,\rho_j\in R^+$

$\left(i=1,2,.......\text{p};j=1,2,.......\text{q}\right);\text{Re}\left(\sum_{u=1}^{r}\theta_u\right)>\text{Re}\left(\sum_{v=1}^{k}\lambda_v\right)>0;\left|\frac{t}{a}\right|<1$ where ${}_pF_q$ in the generalized hypergeometric function defined in (2.5)

$$\sum_{l=0}^{\infty}\frac{\prod_{u=1}^{r}(\theta_u)_l}{\prod_{v=1}^{k}(\lambda_v)_l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\sum_{u=1}^{r}\theta_u-\sum_{v=1}^{k}\lambda_v+l,a\right)\frac{t^l}{l!}$$

$$=\sum_{i=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\frac{1}{(a+n)^{\sum_{u=1}^{r}\theta_u-\sum_{v=1}^{k}\lambda_v}}\,{}_uF_v\left[\begin{matrix}(\theta_u);\ \dfrac{t}{}\\(\lambda_v);\ a+n\end{matrix}\right]\qquad ...(3.7)$$

The proof of Theorem (3.2),(3.3),(3.4),(3.5) are developed following the similar lines as to proof of (2.1),(2.2),(2.3) and(2.4) respectively in view of the definition of general function defined in (3.1)

## 4. Applications

The functions studied in this paper are very general in nature and unify all the functions related to Hurwitz-Lerch Zeta function, therefore many known and New results can be obtained from our main results. As an application, some of these have been shown to derive as follows.

(i) If in the result (1.17), we consider $Lim\ \alpha\to 0$, then it reduces to the known result due to Srivastava et al [5, p. 494,eq. (2.4)]
(ii) If in result (2.1) to (2.4), we apply $Lim\ \alpha\to 0$, then these results reduce to the known result studied by Gupta [12, pp. 132-133, eq. (2.9.1) to (2.9.4)]respectively.
(iii) If in result (3.2) we take $Lim\ \alpha\to 0$, then it reduces to the integral representation for Hurwitz-Lerch Zeta function studied by Srivastava et al [5, pp.504, eq. (6.4)]
(iv) If we take $\alpha\to 0$, the result (3.4) to (3.7) reduce to known results for Hurwitz-Zeta function due to Gupta [12, pp. 134-135, eq. (2.9.7) to (2.9.10)]
(v) If we take $\mu=\sigma,\delta=\xi,v=1$ in (2.1), it reduces to the known generating function due to Virendra [11, p. 17, eq. (3.1)] at $\delta=1$.

(vi) For $p=2, q=1, \mu=\sigma, v=1, \delta=\xi$ the theorem 2.4 provides the known results due to Virendra [11, p. 17, Theorem (3.2)].

## 5. Conclusion

The general function and its Mild generalization studied in this paper unify all the function related to Hurwitz-Lerch Zeta function, its extension as well as the Mittag-Leffler function and their extension scattered in the literature. Many properties involving these general functions like The derivatives, the fractional calculus and fractional order differential equations and more results related to finite and infinite integrals can be established. The present study is worthwhile as the properties of Mittage-Leffler function and their generalizations are being study [18,19,20]. The Mittag-Leffler function is widely used to solve differential equation of fractional order [21,22]. So our proposed study will be very useful to the young researchers, who are engaged in the field of study.

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Panamerican Mathematical Journal
ISSN: 1064-9735
Vol. 35 No. 4s (2025)

# Certain Integral Representation and Fractional Derivatives Associated to a General Function Related to Hurwitz-Lerch Zeta Function

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***Article History:***
**Received**: 4 Mar 2025
**Revised**: 6 May 2025
**Accepted**: 17 Jun 2025

**Abstract:**
In this paper, some integral representation and fractional derivatives of a general function are established. The general function studied in this paper unifies the Mittag-Leffler function and the Hurwitz-Lerch Zeta function. We have evaluated certain type of integrals involving the Mittag-Leffler function which gives the general function as resultant. As application of these results the integral representation and fractional derivatives for simpler function related to the Hurwitz-Lerch Zeta function are shown to be derived.

**Keywords:** Hurwitz-Lerch Zeta function, Mittag-Leffler function, Integral representation, Riemann-Liouville fractional derivatives.

**MSC 2020:** Primary 11M35; 33E20; 26A33 Secondary: 33E12

## 1. Introduction

A general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in the following form [1, p.27, eq.(1.11.1)]:

$$\phi(z,\mathrm{s},a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \tag{1.1}$$

$$\left(a\in C/Z_0^-, \mathrm{s}\in C \text{ when } |z|<1 \text{ and } \mathrm{Re}(s)>1 \text{ when } |z|=1\right)$$

The Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in (1.1) have a special case, as the Riemann Zeta function $\zeta(s)$ and Lerch Zeta function $\zeta(s,a)$ defined by following manner [1, Chapter-1] [2, Chapter-2]

$$\zeta(s)=\sum_{n=0}^{\infty}\frac{1}{(n+1)^s}=\phi(1,s,1) \quad (\mathrm{Re}(s)>1) \tag{1.2}$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\phi(1,s,a) \qquad \left(\mathrm{Re}(s)>1; a\in C/Z_0^-\right) \tag{1.3}$$

An extension of the Hurwitz-Lerch Zeta function was investigated and studied by Lin and Srivastava in following manner [3, p.727, eq.(8)]

$$\phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s} \tag{1.4}$$

$(\mu\in C; a,\nu\in C/Z_0^-, \rho,\sigma\in R^+; \rho<\sigma$ when $s,z\in C; \rho=\sigma$ and $s\in C$ when $|z|<\delta=\rho^{-\rho}\sigma^{\sigma}; \rho=\sigma$ and $\mathrm{Re}(s-\mu-\nu)>1$ when $|z|=\delta)$

where $(\lambda)_n$ denotes the general pochhammer symbol which is defined by

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma\gamma} \tag{1.5}$$

$$(\gamma)_0=1,(\gamma)_n=\gamma(\gamma+1).....(\gamma+n-1)$$

where $n=1,2,....$

We can find from (1.4) that

$$\phi_{(\mu,1)}^{(1,1)}(z,s,a)=\phi_{\mu}^{*}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(a+n)^s} \tag{1.6}$$

$(\mu\in C; a\in C/Z_0^-, s\in C$ when $|z|<1$ and $\mathrm{Re}(s-\mu)>1$ when $|z|=1)$

Where $\phi_{\mu}^{*}(z,s,a)$ is the Hurwitz-Lerch Zeta function considered by Goyal and Laddha [4, p.100, eq.(1.5)].

A generalization of Hurwitz-Lerch Zeta function (1.1) and (1.6) was defined and investigated by Garg et. al. [5, p.313, eq.(1.7)]:

$$\phi_{\lambda,\mu,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n\, n!}\frac{z^n}{(n+a)^s} \tag{1.7}$$

$(\lambda,\mu\in C; \nu,a\in C/Z_0^-; s\in C$ when $|z|<1$ and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=1)$

A further generalization and investigation of a new family of Hurwitz-Lerch Zeta function defined by Srivastava and Pogany [6, p.491, eq.(1.20)] in the following form:

$$\phi_{\lambda,\mu,\nu}^{\rho,\sigma,\mathrm{k}}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}\, n!}\frac{z^n}{(a+n)^s} \tag{1.8}$$

$(\lambda,\mu\in C;\ \nu,a\in C/Z_0^-;\ \rho,\sigma,\mathrm{k}\in\mathrm{R}^+;\ k-\rho-\sigma>-1$ when $s,z\in C; k-\rho-\sigma=-1$ and $s\in C$ when $|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}\mathrm{k}^k;\ k-\rho-\sigma=-1$ and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=\delta^*)$

The well known Mittag-Leffler function $E_\alpha(z)$ is defined by following form [19] (See also [20]):

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ (\mathrm{z}\in C,\mathrm{Re}(\alpha)>0) \tag{1.9}$$

A generalization of Mittag-Leffler function is introduced by Prabhakar [21] as follows:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!} \tag{1.10}$$

where $\alpha,\beta,\gamma\in C;\operatorname{Re}(\alpha)>0;\operatorname{Re}(\beta)>0;z\in C$

This function further generalized by Srivastava and Tomovski [22] as following form (see also [23],[24]):

$$E_{\alpha,\beta}^{\gamma,k}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{1.11}$$

where $z,\beta,\gamma,k\in C;\operatorname{Re}(\alpha)>max\{0,\operatorname{Re}(k)-1\};\operatorname{Re}(k)>0$

In view of extension of parameters, Mittag-Leffler function was also studied by Khan et. al. [7, p.2, eq.(1.9)]:

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\eta,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}(\delta)_{pn}}\frac{z^n}{\Gamma(\alpha n+\beta)} \tag{1.12}$$

where $\alpha,\beta,\eta,\delta,\mu,\nu,\rho,\sigma\in C;p,q>0$ and $q\le\operatorname{Re}(\alpha)+p$ and

$\min\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\eta),\operatorname{Re}(\delta),\operatorname{Re}(\mu),\operatorname{Re}(\nu),\operatorname{Re}(\rho),\operatorname{Re}(\sigma)\}>0$

The function (1.12) at $\delta=p=1$ takes the following form

$$E_{\alpha,\beta,\nu,\sigma}^{\mu,\rho,\eta,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}z^n}{(\nu)_{\sigma n}n!\Gamma(\alpha n+\beta)} \tag{1.13}$$

where $\alpha,\beta,\eta,\nu,\rho,\sigma\in C;$ and $\min\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\eta),\operatorname{Re}(\nu),\operatorname{Re}(\rho),\operatorname{Re}(\sigma)\}>0.$

In 1940, Wright Introduced and studied the following Taylor-Maclaurin Series [8, p.424]:

$$\in_{\alpha,\beta}(\phi;z)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{\Gamma(\alpha n+\beta)} \tag{1.14}$$

$$(\alpha,\beta\in C;\operatorname{Re}(\alpha)>0)$$

where $\phi(n)$ is a function satisfying suitable conditions.

Motivated by the work of Wright [8], E.W. Barnes [16] considered the asymptotic expansion of function in the class which is defined as:

$$E_{\alpha,\beta}^{(k)}(s,z)=\sum_{n=0}^{\infty}\frac{z^n}{(n+k)^s\Gamma(\alpha n+\beta)} \tag{1.15}$$

$$(\alpha,\beta\in C;\operatorname{Re}(\alpha)>0)$$

for suitable restricted parameters $k$ and $s$.

Recently for suitably restricted sequence $\{\phi(n)\}_{n=0}^{\infty}$, Srivastava [9] introduced a class of function (See also [10],[11]):

$$\in_{\alpha,\beta}(\phi;z,s,a)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{(a+n)^s\Gamma(\alpha n+\beta)} \tag{1.16}$$

$$(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0)$$

Further for $\phi(n)=\frac{(\mu)_n}{n!}$ ,the following function recently studied by Kumar [12, p.13, eq.(1.9)] in the following manner:

$$\in_{\alpha,\beta}^{\mu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(\delta n+a)^s\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{1.17}$$

where $\mathrm{Re}(\mu)>0$ ; $\mathrm{Re}(\delta)>0$ ; $\mathrm{Re}(a)>0$ ; $\mathrm{Re}(s)\geq 0$ ; $\mathrm{Re}(\alpha)>0$ ; $\mathrm{Re}(\beta)>0$ and $|z|\leq 1$.

To give more extension in view of the parameters the following more generalized function defined and represented as below [25, p.954, eq.(1.17)]

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\rho n}(a+n)^s\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{1.18}$$

where $\alpha,\beta,\mu,\lambda\in C$ ; $a,\sigma\in C/Z_0^-$ ; $\delta,\nu,\rho\in R^+$ ; $\rho-\delta-\nu>-1$ when $s,z\in C$ ; $\rho-\delta-\nu=-1$ and $s\in C$ when $|z|<\xi^*=\delta^{-\delta}\nu^{-\nu}\rho^{\rho}$; $\rho-\delta-\nu=-1$ and $\mathrm{Re}(s+\sigma-\mu-\lambda)>1$ when $|z|=\xi^*$.

Through out this paper, we used to the following well known facts and rules.

(i) The Riemann-Liouville fractional derivatives operator $D_z^{\mu}$ is defined by [13 p.181]:

$$D_z^{\mu}\{f(z)\}=\begin{cases}\dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}f(t)\,dt, & (\mathrm{Re}(\mu)<0)\\[2ex] \dfrac{d^m}{dz^m}D_z^{\mu-m}\{f(z)\}, & (m-1\leq\mathrm{Re}(\mu)<m\,(m\in N))\end{cases} \tag{1.19}$$

It is known that

$$D_z^{\mu}\{z^{\lambda}\}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}z^{\lambda-\mu},(\mathrm{Re}(\lambda)>-1) \tag{1.20}$$

(ii) The Gamma integral [14, p.884,eq.312(2)]

$$\frac{\Gamma(\alpha)}{a^{\alpha}}=\int_0^{\infty}x^{\alpha-1}e^{-ax}dx,\mathrm{Re}(\alpha)>0 \tag{1.21}$$

(iii) The Binomial expansion [15, p.20,eq.(29)]

$$(1-x)^{-a}=\sum_{n=0}^{\infty}(a)_n\frac{x^n}{n!} \tag{1.22}$$

where $(a)_n$ is the pochhammer's symbol defined in (1.5).

The integral representation of the function defined in (1.18) is also studied by the authors in [25, p.954, eq.(1.18)] as the following theorem.

**Theorem-1.1**

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \alpha,\beta,\mu,\lambda\in C;\ a,\sigma\in C/Z_0^-;\ \delta,\nu,\rho\in R^+;\ \rho-\delta-\nu\geq -1$ with $|z|<\xi^*=\delta^{-\delta}\nu^{-\nu}\rho^{\rho}$

the integral representation of the function $\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)$ holds true

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)dt \tag{1.23}$$

where $E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittag-Leffler function defined in (1.13).

## 2. Integral Representations And Fractional Derivative

In this section the three integral representations for the function (1.18) and the fractional derivative involving this general function are established.

**Result 2.1**

For $w,\lambda,\mu\in C;\ \nu\in C/Z_0^-;\ p\in N_0;\ \min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \delta,\nu\in R^+;\ \rho-\delta-\nu\geq -1$ with equality when $|z|<\xi^{\rho}=\delta^{-\delta}\nu^{-\nu}\rho^{\rho}$,

we have the following integral representation

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\int_0^\infty t^{s-1}e^{-(a+kp)t}\left(1-e^{-pt}\right)^{w}E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)dt \tag{2.1}$$

provided that each the side of (2.1) exists and $E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittag-Leffler function defined in (1.13).

**Result 2.2**

For $\lambda,\mu\in C;\ \nu\in C/Z_0^-;\ \min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\ \delta,\nu,\rho\in R^+;\ \rho-\delta-\nu\geq -1$ with equality when $|z|<\xi^{\rho}=\delta^{-\delta}\nu^{-\nu}\rho^{\rho}$,

we havethe following integral representation

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\sum_{k=0}^{\infty}\int_0^\infty t^{s-1}e^{-\left(a+\frac{k(k+1)}{2}\right)t}\left(1-e^{-(k+1)t}\right)\ E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)dt \tag{2.2}$$

provided that the each side of (2.2) exists and $E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittag-Leffler function defined in (1.13).

**Result 2.3**

Let $(a_n)_{n\in N_0}$ be a positive sequence such that the following Infinite series $\sum_{n=0}^{\infty} e^{-a_n t}$ converges for any $t \in R^+$ and for $\min\{\mathrm{Re}(a), \mathrm{Re}(s)\} > 0$
we have the following integral representation

$$\in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}(z,s,a) = \frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1} e^{-(a-a_0+a_n)t}\left[1-e^{-(a_{n+1}-a_n)t}\right] E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right) dt \tag{2.3}$$

provided that the each side of (2.3) exists and $E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}(z)$ is the Mittag-Leffler function defined in (1.13).

**Result 2.4**

For $\mathrm{Re}(v) > 0$ and $\rho > 0$, we have the following fractional derivative

$$D_z^{\sigma-\tau}\left\{z^{\sigma-1} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z^{\rho},s,a\right)\right\} = \frac{\Gamma\sigma}{\Gamma\tau} z^{\tau-1} \in_{\alpha,\beta,\tau,\rho}^{\mu,\delta,\lambda,v}\left(z^{\rho},s,a\right) \tag{2.4}$$

provided that each the side of (2.4) exists.

## Outlines Of Proofs

**Proof of Result 2.1**

To prove result (2.1), let the right hand side of (2.1) is denoted by $\Delta_1$ i.e.

$$\Delta_1 = \frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\int_0^{\infty} t^{s-1} e^{-(a+kp)t}\left(1-e^{-pt}\right)^w E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right) dt$$

On changing the order of integration and summation, we have

$$\Delta_1 = \frac{1}{\Gamma s}\int_0^{\infty}\left\{\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\left(e^{-pt}\right)^k\right\} t^{s-1} e^{-at}\left(1-e^{-pt}\right)^w E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right) dt$$

Now using the binomial expansionfrom (1.22), above expression takes the following form

$$\Delta_1 = \frac{1}{\Gamma s}\int_0^{\infty} t^{s-1} e^{-at} E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right) dt$$

Now interpret the above integral in view of (1.23), we arrive at once the desired result in (2.1).

**Proof of Result 2.2**

To prove result (2.2), let the right hand side of (2.2) is denoted by $\Delta_2$ i.e.

$$\Delta_2 = \frac{1}{\Gamma s}\sum_{k=0}^{\infty}\int_0^{\infty} t^{s-1} e^{-\left(a+\frac{k(k+1)}{2}\right)t}\left[1-e^{-(k+1)t}\right] E_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(ze^{-t}\right) dt$$

Using the integral representation (1.23) the above integral can be written as

$$\Delta_2 = \sum_{k=0}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+\frac{k(k+1)}{2}\right) - \sum_{k=0}^{\infty} \in_{\alpha,\beta,\sigma,\rho}^{\mu,\delta,\lambda,v}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

Which gives

$$\Delta_2 = \in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z;s,a)+\sum_{k=1}^{\infty}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}\left(z;s,a+\frac{k(k+1)}{2}\right)-\sum_{k=0}^{\infty}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}\left(z;s,a+(k+1)\left(\frac{k}{2}+1\right)\right)$$

After simplifying, the complete expression becomes $\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z,s,a)$ and we get the required result in (2.2).

**Proof of Result 2.3**

To prove result (2.3), let the right hand side of (2.3) is denoted by $\Delta_3$ i.e.

$$\Delta_3 = \frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty} t^{s-1}e^{-(a-a_0+a_n)t}\left[1-e^{-(a_{n+1}-a_n)t}\right]E^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}\left(ze^{-t}\right)dt$$

Using the integral representation (1.23) it can be written in the following form

$$\Delta_3 = \sum_{n=0}^{\infty}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z,s,a-a_0+a_n)-\sum_{n=0}^{\infty}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z,s,a-a_0+a_{n+1})$$

Which gives

$$\Delta_3 = \in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z;s,a)+\sum_{n=1}^{\infty}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z;s,a-a_0+a_n)-\sum_{n=0}^{\infty}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z,s,a-a_0+a_{n+1})$$

After simplifying, the above complete expression becomes $\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z,s,a)$ and we get the required result in (2.3).

**Proof of Result 2.4**

To prove result (2.4), let the left hand side of (2.4) is denoted by $\Delta_4$ i.e.

$$\Delta_4 = D_z^{\sigma-\tau}\left\{z^{\sigma-1}\in^{\mu,\delta,\lambda,\nu}_{\alpha,\beta,\sigma,\rho}(z^{\rho};s,a)\right\}$$

Using the definition of (1.18) and on simplification we have the following form

$$\Delta_4 = \sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\rho n}(a+n)^s\Gamma(\alpha n+\beta)n!}D_z^{\sigma-\tau}\left\{z^{\rho n+\sigma-1}\right\}$$

Now on using (1.20) and (1.5), itcan be written as follows

$$\Delta_4 = \frac{\Gamma\sigma}{\Gamma\tau}z^{\tau-1}\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\tau)_{\rho n}(a+n)^s\Gamma(\alpha n+\beta)}\frac{(z^{\rho})^n}{n!}$$

On interpreting the series in view of (1.18) we achieve at once the desired result in (2.4).

## 3. Mild Generalization

We also studied here a mild generalization of the function defined in (1.18) in view of parameters. it is defined and represented in the following manner [25, p.958, eq. (3.1)]

$$\in^{(\mu_p);(\delta_p)}_{\alpha,\beta;(\sigma_q);(\rho_q)}(z;s,a)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(a+n)^s\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{3.1}$$

where $p,q \in N_0$; $s,z,\mu_i,\alpha,\beta \in C$; $a,\sigma_j \in C/Z_0^-$; $\delta_i,\rho_j \in R^+$, $\forall(i=1,2,.......p; j=1,2........q)$,

and $(\mu_p) \equiv \mu_1,\mu_2,...,\mu_p$; $(\delta_p) \equiv \delta_1,\delta_2,...,\delta_p$ and similar notation stands for $(\sigma_p)$ and $(\rho_q)$.
The integral representations for the function defined in (3.1) are also recently studied by the authors [25, p.959, eq.(3.2)] as the following result.

**Result 3.1**
For $p,q \in N_0$; $s,z,\mu_i,\alpha,\beta \in C$; $a,\sigma_j \in C/Z_0^-$; $\delta_i,\rho_j \in R^+$, $\forall(i=1,2,.......p; j=1,2........q)$,

we have,

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a) = \frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(ze^{-t})dt \tag{3.2}$$

provided that each the side of (3.2) exists and $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is the Mittag-Leffler function defined as following manner[25, p.959, eq.(3.3)]:

$$E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z) = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}. \tag{3.3}$$

Certain new integral representations for the generalized function defined in (3.1) and the fractional derivative involving this general function are established here as given below.

**Result 3.2**
For all $i=1,2,.......\text{p}$; $j=1,2,.......\text{q}$; $p,q \in N_0$; $s,z,\mu_i \in C$; $a,\sigma_j \in C/Z_0^-$; $\delta_i,\rho_j \in R^+$

$\min\{\text{Re}(a),\text{Re}(s)\} > 0$, we have

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a) = \frac{1}{\Gamma s}\sum_{k=0}^{\infty}\frac{(w)_k}{k!}\int_0^\infty t^{s-1}e^{-(a+kp)t}\left(1-e^{-pt}\right)^w E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(ze^{-t})\,dt \tag{3.4}$$

provided that each the side of (3.4) exists and $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is Mittag-Leffler function definedin (3.3).

**Result 3.3**
For all $i=1,2,.......\text{p}$; $j=1,2,.......\text{q}$; $p,q \in N_0$; $s,z,\mu_i \in C$; $a,\sigma_j \in C/Z_0^-$; $\delta_i,\rho_j \in R^+$

$\min\{\text{Re}(a),\text{Re}(s)\} > 0$, we have

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a) = \frac{1}{\Gamma s}\sum_{k=0}^{\infty}\int_0^\infty t^{s-1}e^{-\left(a+\frac{k(k+1)}{2}\right)t}\left[1-e^{-(k+1)t}\right] E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(ze^{-t})\,dt \tag{3.5}$$

provided that each the side of (3.5) exists and $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is Mittag-Leffler function definedin (3.3).

**Result 3.4**

For all $i=1,2,.......p$ ; $j=1,2,.......q$ ; $p,q\in N_0$ ; $s,z,\mu_i\in C$ ; $a,\sigma_j\in C/Z_0^-$ ; $\delta_i,\rho_j\in R^+$

$\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0$. Also let $(a_n)_{n\in N_0}$ be the positive sequence such that the following infinite

series $\sum_{n=0}^{\infty}e^{-a_n t}$ converges for any $t\in R^+$ then

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z;s,a)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\int_0^{\infty}t^{s-1}e^{-(a-a_0+a_n)t}\left[1-e^{-(a_{n+1}-a_n)t}\right]\ E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt \quad (3.6)$$

provided that each the side of (3.6) exists and $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is Mittag-Leffler function defined in (3.3).

**Result 3.5**

For $\mathrm{Re}(v)>0;\lambda>0$ we have the following fractional derivative

$$D_z^{v-\tau}\left\{z^{v-1}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_q)}\left(z^{\lambda};s,a\right)\right\}=\frac{\Gamma v}{\Gamma\tau}z^{\tau-1}\in_{\alpha,\beta;(\sigma_q),\tau,(\rho_q),\lambda}^{(\mu_p),v;(\delta_q),\lambda}\left(z^{\lambda};s,a\right) \quad (3.7)$$

provided that each the side of (3.7) exists.

The proofs of Results 3.2, 3.3, 3.4 and 3.5 follow the similar lines as to prove the results in equations (2.1),(2.2),(2.3) and (2.4) respectively in view of the definition of generalized function (3.1).

## 4. Certain Known Special Cases

(i) For limit $\alpha\to 0$ the results (1.23), (2.3), (2.4), (3.6), (3.7) reduce respectively to the known results due to Srivastava et al.[6, p.494, eq.(2.4); p.496, eq.(2.18); p.502, eq.(5.1); p.505, eq.(6.9); p.505, eq.(6.8)].

(ii) If in results (2.1) and (2.2), on letting limit $\alpha\to 0$ these reduce to the known results due to Gupta ([17], pp. 188-189, eq. (3.9.1), (3.9.2) at b=0).

(iii) If we take limit $\alpha\to 0$ and $\delta=v=\rho=1$ in (2.1) and (2.2) then these reduce to the known results due to Garg et al [5, p. 314, eqs. (2.4), (2.3)] which in turn at $\mu=\sigma,\lambda=1$ gives the results [18, p.129, eqs. (46),(47)].

(iv) If we consider limit $\alpha\to 0$ and $\delta=v=\rho=1$ , $\lambda=\sigma$ in (2.4) then it reduce to the known result due to Gupta [17, p.149, eqs.(3.4.7)].

(v) If we take limit $\alpha\to 0$ and $\delta=v=\rho=1,\tau=\mu$ in (2.4) then it reduce to the known result due to Gupta [17, p.149, eqs.(3.4.8)].

(vi) For limit $\alpha\to 0$, the Result 3.5 reduce to the known result due to Gupta ([17], p.175, eqs.(3.8.16) at $b=0$ and $\beta=1$).

## 5. Conclusion

The integral representation and fractional derivatives of a general function studied by Srivastava [9] and Kumar [12] are investigated in this paper. The series integral representation involving Mittag-Leffler function are evaluated in term of this general function. As this general function unifies many functions related to Hurwitz-Lerch Zeta function and Mittag-Leffler function. Therefore many integral representations and fractional derivatives of simpler function can be derived. We have given few known results obtained as special cases in Section-4 from our main results studied in this paper. Recently the authors [25] established certain generating functions for the function defined in (1.18) and it's mild extension defined in (3.1). We refer [25] to the interested readers. These results are very useful and provide new direction and further scope of research for the young researchers engaged in the field.

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